Klt varieties with trivial first Chern class

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Grenoble, 2-3 July 2019

1 Introduction

The goal of these lectures is to explain the rough scheme of the proof of the celebrated decomposition theorem due to Beauville and Bogomolov, about compact Kähler manifolds with zero first Chern class. Then, I will explain the following result which is a key ingredient to prove the generalization of the decomposition theorem for projective klt varieties with zero first Chern class.

Theorem 1.1 (GGK)**.** *Let X be a projective variety with klt singularities and $K_X \equiv 0$. Then, there exists a finite, quasi-étale map $f : X' \to X$, an abelian variety A, a projective variety Z with canonical singularities and $K_Z \sim 0$ such that $X' \simeq A \times Z$ and such that the tangent sheaf of Z decomposes as*

$$\mathscr{T}_Z \simeq \bigoplus_{i \in I} \mathscr{E}_i$$

where $\mathscr{E}_i$ are foliations of rank at least two, with trivial determinant and such that $\mathrm{Sym}^{[k]}\mathscr{E}_i$ is strongly stable for any integer $k \geqslant 1$.

2 Ricci curvature, holonomy and stability

2.1 Yau's Theorem

Let X be a complex manifold of dimension n.

Definition 2.1. A Kähler metric ω is a smooth, real $(1,1)$-form such that ω is *closed* and *positive*. In local coordinates, the positivity condition means that if one writes

$$\omega = \sum_{\alpha,\beta} \omega_{\alpha\bar{\beta}}\, i dz_\alpha \wedge d\bar{z}_\beta \tag{2.1}$$

then the $n \times n$ complex matrix $(\omega_{\alpha\bar{\beta}})$ is hermitian definite positive.

Definition 2.2. Let ω be a Kähler metric that we write locally as in (2.1). Then, the locally defined $(1,1)$-form $-i\partial\bar{\partial}\log\det(\omega_{\alpha\bar{\beta}})$ is a globally well-defined closed form denoted by $\mathrm{Ric}(\omega)$. By abuse of notation, one writes

$$\mathrm{Ric}(\omega) = -i\partial\bar{\partial}\log\omega^n.$$

Moreover, its cohomology class $[\mathrm{Ric}(\omega)] \in H^{1,1}(X,\mathbb{R})$ is independent of ω and is equal to $-c_1(K_X)$.

Theorem 2.3 (Yau '78)**.** *Let X be a compact Kähler manifold and let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a Kähler class. Assume that $c_1(K_X) = 0 \in H^{1,1}(X,\mathbb{R})$.*
Then, there exists a unique Kähler metric $\omega \in \alpha$ such that $\mathrm{Ric}(\omega) = 0$.

2.2 Holonomy

Definition 2.4. Let (X, ω) be a connected Kähler manifold and let $x \in X$. Let D be the Chern connection associated to ω. It can be seen as an operator

$$\begin{aligned} D : \mathcal{C}^\infty(X, T_X) &\to \mathcal{C}^\infty(X, \mathrm{End}(T_X)) \\ u &\mapsto (v \mapsto D_u v) \end{aligned}$$

Given a loop $\gamma : [0,1] \to X$ based at x and given a tangent vector $v \in T_{X,x}$, there exists a unique smooth section $v(t)$ of $\gamma^* T_X$ such that

$$\begin{cases} v(0) = v \\ D_{\dot{\gamma}(t)} v(t) = 0 \end{cases}$$

The vector $\tau_\gamma(v) := v(1)$ is called the parallel transport of v along γ. This defines an isometry τ_γ of $(T_{X,x}, \omega_x)$ that one sees as an element of $\mathrm{U}(n)$ (we implicitly use the Kähler condition).

Definition 2.5. The holonomy group (resp. restricted holonomy group) of (X, ω) at x is defined by

$$\begin{aligned} \mathrm{Hol}(X, \omega) &= \{\tau_\gamma, \gamma \text{ loop at } x\} \subset \mathrm{U}(n) \\ \mathrm{Hol}^\circ(X, \omega) &= \{\tau_\gamma, \gamma \text{ loop at } x \text{ homotopic to zero}\} \subset \mathrm{U}(n) \end{aligned}$$

It is easy to see that they are indeed groups. In the following, one sets $G := \mathrm{Hol}(X, \omega)$ and $G^\circ := \mathrm{Hol}^\circ(X, \omega)$. With these notations, G° is the connected component of the identity of G, and there exists a tautological surjection

$$\pi_1(X) \twoheadrightarrow G/G^\circ.$$

Definition 2.6. With the notations above, the group G acts on $T_{X,x} \simeq \mathbb{C}^n$ linearly by isometries and induces a semi-simple complex n-dimensional representation of G called the holonomy representation.

Proposition 2.7 (Holonomy principle)**.** *Let (X, ω, x) and G as above. The evaluation map ev_x induces a $1:1$ correspondence*

$$\begin{aligned} \{\sigma \in \mathcal{C}^\infty(X, T_X)\,;\, D\sigma = 0\} &\longrightarrow \{\sigma_x \in T_{X,x}^G\} \\ \{F \subset T_X \text{ complex subbundle}\,;\, D(F) \subset F\} &\longrightarrow \{V \subset T_{X,x} \text{ complex subspace stable under } G\} \end{aligned}$$

The condition $D\sigma = 0$ means that for any $u \in \mathcal{C}^\infty(X, T_X)$, one has $D_u \sigma = 0$. Similarly, $D(F) \subset F$ means that for every $u \in \mathcal{C}^\infty(X, T_X)$ and $v \in \mathcal{C}^\infty(X, F)$, one has $D_u v \in \mathcal{C}^\infty(X, F)$.

Remark 2.8. The same principle holds for tensor bundles $E = T_X^{\otimes p} \otimes (T_X^*)^{\otimes q}$ by considering the induced connection D on E and the induced tensor representation of G on E_x.

Definition 2.9. Let (X, ω) as above and let $E := T_X^{\otimes p} \otimes (T_X^*)^{\otimes q}$. Let $\sigma \in \mathcal{C}^\infty(X, E)$ and let $F \subset E$ be a complex subbundle. One says that σ (resp. F) is a parallel tensor (resp. subbundle) if $D\sigma = 0$ (resp. $D(F) \subset F$).

Proposition 2.10. *Let (X, ω) as above and let $E := T_X^{\otimes p} \otimes (T_X^*)^{\otimes q}$. Let $\sigma \in \mathcal{C}^\infty(X, E)$ and let $F \subset E$ be a complex subbundle.*

1. *If σ is parallel, then σ is holomorphic (i.e. $D\sigma = 0 \implies \bar{\partial}\sigma = 0$).*
2. *If F is parallel, then it is a holomorphic subbundle and the $\mathcal{C}^\infty$ splitting $E = F \oplus F^\perp$ is holomorphic.*

Theorem 2.11 (Bochner's principle). *Let (X, ω) be a compact Kähler manifold such that* $\mathrm{Ric}(\omega) = 0$ *and let $\alpha = [\omega] \in H^{1,1}(X, \mathbb{R})$.*
Then, any holomorphic tensor $\sigma \in H^0(X, E)$ is parallel.

Remark 2.12. From the assumptions, it follows that $c_1(E) = (q-p)c_1(K_X) = 0$.

Proof. Apply Bochner's identity

$$\Delta_\omega |\sigma|^2_\omega = |D\sigma|^2_\omega$$

and integrate against ω^n. □

As a consequence, one has a $1:1$-correspondence

$$\{\sigma \in H^0(X,E)\} \longleftrightarrow \{\sigma_x \in E_x^G\}$$

2.3 Stability

Let X be a Kähler manifold of dimension n, let E be a holomorphic vector bundle on X and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a Kähler class.

Definition 2.13. One says that E is stable (resp. semistable) with respect to α if for any proper non-zero subsheaf $\mathcal{F} \subset \mathcal{O}_X(E)$, one has

$$\frac{c_1(\mathcal{F}) \cdot \alpha^{n-1}}{\mathrm{rk}(F)} < \frac{c_1(E) \cdot \alpha^{n-1}}{\mathrm{rk}(E)} \quad (\text{resp. } \leqslant \text{ }).$$

One says that E is polystable if E is the direct sum of stable subbundles of the same slope as E, or equivalently if E is semistable and it is the direct sum of stable subbundles.

Theorem 2.14. *Let X be a compact Kähler manifold such that $c_1(K_X) = 0$, let $\alpha = [\omega] \in H^{1,1}(X, \mathbb{R})$ and let $\omega \in \alpha$ be the Kähler Ricci flat metric.*
Then, T_X is polystable with respect to α. More precisely, T_X admits a direct sum decomposition

$$T_X = \bigoplus_{i \in I}^{\perp} E_i$$

into stable, parallel subbundles with $c_1(E_i) = 0$.

Idea of proof. The Ricci flat Kähler metric ω induces a hermitian metric h on T_X which is Hermite-Einstein, i.e. $i\Theta(T_X, h) \wedge \omega^{n-1} = 0$. The induces metric $h|_F$ on a subbundle F satisfies

$$i\Theta(F, h|_F) \wedge \omega^{n-1} = \mathrm{tr}_{\mathrm{End}}(\beta^* \wedge \beta) \wedge \omega^{n-1}$$

where $\beta \in \mathcal{C}^\infty(X, \Omega_X^{1,0} \otimes \mathrm{Hom}(F^\perp, F))$ is the second fundamental form of $(F, h|_F) \subset (T_X, h)$. So the quantity above is non-positive and zero everywhere on X iff the splitting $T_X = F \oplus F^\perp$ is holomorphic. □

Corollary 2.15. *With the notation of the Theorem above, let G be the holonomy group of (X, ω).*
Then, T_X is stable with respect to α if and only if the holonomy representation is irreducible.

Proof. □

Remark 2.16. Both results above hold when replacing T_X by a tensor bundle $E = T_X^{\otimes p} \otimes (T_X^*)^{\otimes q}$ and considering the tensor representation of G induced on E_x.

3 The Decomposition Theorem

3.1 Varieties with trivial first Chern class

Let X be a projective manifold. One defines $K_X := \det(\Omega_X)$, this is a line bundle. One can look at manifolds satisfying the increasingly weaker conditions below

1. $K_X \sim 0$ in $\mathrm{Pic}(X)$
   $\iff \exists\, \Omega \in H^0(X, K_X)$ such that Ω never vanishes. Equivalently, $\Omega \wedge \overline{\Omega}$ is a volume form.
2. K_X is torsion in $\mathrm{Pic}(X)$, that is, there exists $m \in \mathbb{N}^*$ such that $K_X^{\otimes m} \sim 0$.
   $\iff \exists\, \Omega \in H^0(X, K_X^{\otimes m})$ such that Ω never vanishes. Equivalently, $(\Omega \wedge \overline{\Omega})^{1/m}$ is a volume form.
3. $c_1(K_X) = 0$ in $H^2(X, \mathbb{R})$.
   $\iff \exists\, \omega$ Kähler form on X, there exists $f_\omega \in \mathcal{C}^\infty(X)$ such that $\mathrm{Ric}(\omega) = i\partial\bar{\partial} f_\omega$.
   $\iff \forall\, \omega$ Kähler form on X, there exists $f_\omega \in \mathcal{C}^\infty(X)$ such that $\mathrm{Ric}(\omega) = i\partial\bar{\partial} f_\omega$.

Obviously, $1 \implies 2 \implies 3$. Moreover, it is easy to check that 2 is equivalent to asking that there exists a finite étale morphism $f : Y \to X$ such that $K_Y \sim 0$.

Example 3.1. According to the dimension, one has

1. In dimension one, tori/elliptic curves $\mathbb{C}/\Lambda$ where Λ is a lattice.
2. In dimension two, one has tori, $K3$ surfaces (simply connected surfaces with trivial canonical bundle), Enriques surfaces $K3/\langle \iota \rangle$ where ι is a fixed-point free involution.
3. In higher dimension, tori, products, hypersurfaces $X_d \subset \mathbb{P}^n$ of degree $d = n + 1$.

3.2 The Decomposition Theorem

Theorem 3.2 (Beauville-Bogomolov)**.** *Let X be a projective manifold (or compact Kähler manifold) such that $c_1(K_X) = 0$. Then, there exists a finite étale cover $f : X' \to X$ such that*

$$X' \simeq A \times \prod_{j \in J} Y_j \times \prod_{k \in k} Z_k$$

where A is an abelian variety (or a torus), Y_j is a Calabi-Yau manifold and Z_k is an irreducible holomorphic symplectic manifold. Moreover, the decomposition of X' is unique up to permutation of the factors.

Definition 3.3. Let X be a *simply connected* projective manifold (or compact Kähler manifold) of dimension n. One says that

1. X is a Calabi-Yau (CY) manifold if there exists a nowhere vanishing holomorphic n-form Ω such that one has an algebra isomorphism

$$\bigoplus_{p=0}^{n} H^0(X, \Omega_X^p) = \mathbb{C}[\Omega].$$

2. X is an irreducible holomorphic symplectic (IHS) manifold if there exists a nowhere degenerate holomorphic 2-form σ such that one has an algebra isomorphism

$$\bigoplus_{p=0}^{n} H^0(X, \Omega_X^p) = \mathbb{C}[\sigma].$$

   In particular, n is even and $\sigma^{n/2}$ is a nowhere vanishing holomorphic n-form.

Remark 3.4. A byproduct of the Decomposition theorem is that $c_1(K_X) = 0$ implies that K_X is torsion and that $\pi_1(X)$ is virtually abelian.

3.3 Main Steps of the Proof

Step 1. [Yau]
Find a Kähler metric ω such that $\mathrm{Ric}(\omega) = 0$.

Step 2. [de Rham]
Look at the universal cover $\widetilde{X}$ of X and split

$$(\widetilde{X}, \widetilde{\omega}) \simeq (\mathbb{C}^r, \omega_{\mathrm{eucl}}) \times \prod_{i \in I} (M_i, \omega_i).$$

Step 3. [Cheeger-Gromoll]
Show that the M_i are compact.

Step 4. [Berger-Simons]
Classify the (M_i, ω_i) in terms of their holonomy: it is either SU or Sp.

Step 5. [Bochner]
Translate the holonomy condition into an intrinsic geometric property (existence of holomorphic differential forms) $\rightsquigarrow$ M_i is either a CY or an IHS. In particular $H^0(M_i, T_{M_i}) = 0$.

Step 6. [Bieberbach]
Find a normal subgroup of finite index $G < \pi_1(X)$ such that

$$G \backslash \widetilde{X} \simeq \mathbb{C}^r / \Lambda \times \prod_{i \in I} M_i.$$

4 The singular case

In the following, X will be a *projective* variety with klt singularities and trivial first Chern class. Without loss of generality, one can actually assume that X has canonical singularities and trivial canonical bundle. The starting point is the following theorem

Theorem 4.1 (EGZ). *In the context above, let H be an ample Cartier divisor. Then, there exists a unique closed, positive $(1,1)$ current $\omega \in c_1(H)$ such that*

$$\begin{cases} \omega \text{ is a smooth Kähler Ricci flat metric on } X_{\mathrm{reg}}. \\ \int_{X_{\mathrm{reg}}} \omega^n = c_1(H)^n. \end{cases}$$

However, Step 2 and 3 above fail completely in the singular case by lack of completeness of $\omega|_{X_{\mathrm{reg}}}$. The foliations provided by the holonomy decomposition of $(X_{\mathrm{reg}}, \omega|_{X_{\mathrm{reg}}})$ may thus not be easily integrated but they can still be understood at least on an appropriate finite cover. This is the content of the following

Theorem 4.2 (GGK). *Let X be a projective variety with klt singularities and $K_X \equiv 0$. Then, there exists a finite, quasi-étale map $f : X' \to X$, an abelian variety A, a projective variety Z with canonical singularities and $K_Z \sim 0$ such that $X' \simeq A \times Z$ and such that the tangent sheaf of Z decomposes as*

$$\mathscr{T}_Z \simeq \bigoplus_{i \in I} \mathscr{E}_i$$

where the subbundles $\mathscr{E}_i|_{Z_{\mathrm{reg}}} \subset \mathscr{T}_{Z_{\mathrm{reg}}}$ are parallel with respect to any singular Ricci flat metric, and their holonomy is either SU *or* Sp.

Idea of the proof. There are several steps.

- Prove that $\mathscr{T}_X$ is metrically polystable.
- Use Druel's result to split off an abelian variety and end up with a variety Z whose tangent sheaf has no flat summand.
- Pick a singular KE metric ω on Z and look at the holonomy representation $\mathrm{Hol}(Z_{\mathrm{reg}}, \omega|_{X_{\mathrm{reg}}})$ on $T_{Z,z}$. It decomposes into irreducible pieces, yielding a decomposition of $\mathscr{T}_Z$ into parallel subbundles but the holonomy of the various factors is not covered by Berger-Simons list. The issue is that one only knows G° (it is a product of irreducible SU or Sp) but G/G° might be infinite.
- Prove that G/G° is finite, and take the corresponding cover to make the holonomy connected.

Take the example where $\mathscr{T}_X = \mathscr{E}_1 \oplus^\perp \mathscr{E}_2$ where $\mathrm{Hol}^\circ(X_{\mathrm{reg}}, \omega) = \{\mathrm{Id}_{\mathbb{C}^{n_1}}\} \times \mathrm{SU}(n_2)$. As $\mathscr{E}_1$ is flat, by Druel's result, there exists an abelian variety, a klt variety Z and a finite étale cover $f : A \times Z \to X$ such that $\mathscr{T}_{X/A} \simeq f^*\mathscr{E}_2$ and $f^*\omega = \omega_A \oplus \omega_Z$. In particular, $\mathrm{Hol}^\circ(Z_{\mathrm{reg}}, \omega_Z) = \mathrm{SU}(n_2)$. Then $\pi_1(Z_{\mathrm{reg}}) \twoheadrightarrow G_Z/G_Z^\circ \subset \mathbb{U}(1)$. Up to taking a further cover of Z, one can assume that the representation factors through $\pi_1(Z)$ (GKP), hence through $H^1(X, \mathbb{Z})$. As $\dim_{\mathbb{C}} H^1(X, \mathbb{Z}) \otimes \mathbb{C} = 2 \dim_{\mathbb{C}} H^0(X, \Omega_X^{[1]}) \leqslant \dim(\mathbb{C}^{n_2})^{\mathrm{SU}(n_2)} = 0$, the image of the representation is finite. □

Corollary 4.3 (Bochner principle)**.** *Any holomorphic tensor*

$$\sigma \in H^0(X_{\mathrm{reg}}, \mathscr{T}_X^{\otimes p} \otimes (\mathscr{T}_X^*)^{\otimes q})$$

is parallel with respect to any singular Ricci-flat metric.

Idea of the proof. A holomorphic tensor generates a trivial saturated subsheaf $\mathscr{L}$ of the polystable sheaf $\mathscr{T}_X^{\otimes p} \otimes (\mathscr{T}_X^*)^{\otimes q}$. Hence that line is invariant under the holonomy group. As the holonomy does not have any non-trivial character, the pointwise action of the holonomy on the line $\mathscr{L}_x$ is trivial. □

Corollary 4.4. *The sheaves $\mathscr{E}_i$ above are strongly stable; i.e. for any $g : Z' \to Z$ quasi-étale and finite and for any polarization H' on Z', the sheaf $g^{[*]}\mathscr{E}_i$ is stable with respect to H'. More generally, the same is true for any reflexive symmetric power* $\mathrm{Sym}^{[k]}\mathscr{E}_i$, *$k \geqslant 1$.*

Proof. We have seen that stability is equivalent to irreducibility of the holonomy representation. After passing to a quasi-étale, the restricted holonomy does not change. Hence the holonomy does not change either as it is already connected. □

Assuming that X or a cover splits as a product of varieties $\prod X_i$ where $\mathscr{E}_i$ becomes isomorphic to $p_i^*\mathscr{T}_{X_i}$, one could classify the factors X_i in terms of their holomorphic forms.

Corollary 4.5. *If $\mathscr{T}_X$ is stable and remains stable after quasi-étale finite covers, then the algebra of reflexive holomorphic forms on X and any of its covers is either the one of a CY or a IHS.*