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Métriques de Kähler-Einstein singulières

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TABLE DES MATIÈRES

Résumé. Dans cette thèse, la notion centrale est celle de métrique de Kähler-Einstein. À la suite des travaux fondateurs de Aubin et Yau (entre autres) résolvant le problème de l'existence de métrique kähleriennes à courbure de Ricci constante négative ou nulle, les mathématiciens se sont intéressés à diverses généralisations possibles de ces résultats. Nous en présenterons deux aspects ainsi qu'une application en géométrie algébrique. Tout d'abord, nous montrerons l'existence de métriques de Kähler-Einstein à singularités prescrites (de type coniques, ou Poincaré) le long d'un diviseur à croisement normaux. Ensuite, nous construirons des métriques de Kähler-Einstein sur des variétés singulières, à singularités semi-log canoniques (travail en commun avec Robert Berman) ; nous analyserons également les singularités de ces métriques associées à des paires Kawamata log terminales (klt) sur le lieu log-lisse, ce qui fera un lien avec la première partie. Enfin, nous mentionnerons une application de ces techniques à la semi-stabilité ou semipositivité générique du faisceau tangent de variétés à singularités log canoniques.

CHAPITRE 1

PRÉSENTATION DES RÉSULTATS

1.1. Introduction : le problème de Calabi

1.1.1. Métriques de Kähler-Einstein. —

1.1.1.1. Approche du problème. — Etant donnée une variété kählerienne compacte X, peut-on trouver des métriques kähleriennes ω à courbure de Ricci constante, ie

$$(\mathbf{KE}) \qquad \operatorname{Ric}\omega = \lambda\omega$$

pour un certain $\lambda \in \mathbb{R}$?

Une telle métrique est appelée *métrique de Kähler-Einstein*, tout simplement car il s'agit d'une métrique kählerienne qui induit une métrique d'Einstein sur la variété réelle sous-jacente. Rappelons qu'une métrique kählerienne ω est une forme *fermée* de type $(1,1)$ (ie $d\omega = 0$) associée à une métrique hermitienne sur X (ie une métrique riemannienne g sur X vue comme variété réelle, compatible avec la structure complexe J, au sens où J est orthogonal pour g).

La question de l'existence de métriques de Kähler-Einstein est l'une des plus fondamentales en géométrie differentielle (kählerienne). Il est facile de voir qu'il existe des obstructions à l'existence de telles métriques. En effet, quelle que soit la métrique kählerienne ω, la forme de Ricci $\operatorname{Ric}\omega$ définie localement par $i\partial\bar\partial(\log\omega^n)$ appartient toujours à la classe de cohomologie $c_1(K_X^{-1}) \in H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z})$ égale à la première classe de Chern du fibré anticanonique $K_X^{-1} = \Lambda^n T_X$, où n est la dimension complexe de X. Autrement dit la classe de cohomologie $[\operatorname{Ric}\omega] \in H^{1,1}(X,\mathbb{R})$ ne dépend pas de la métrique ω.

Ainsi, s'il existe une métrique de Kähler-Einstein ω vérifiant $\operatorname{Ric}\omega = \lambda\omega$, alors nécessairement, $\lambda[\omega] \in c_1(K_X^{-1})$. Donc la classe $c_1(K_X)$ a un signe (qui est celui de λ), ce qui n'est a priori pas le cas d'une classe quelconque $\alpha \in H^{1,1}(X,\mathbb{R})$.

Il faut donc à l'avance supposer que la classe de cohomologie $c_1(K_X) = -c_1(K_X^{-1})$ a un signe. Si celui-là est non nul, alors le théorème de Kodaira impose que la variété kählerienne compacte X soit en faite projective, c'est-à-dire qu'elle admet un plongement dans un espace projectif $\mathbb{P}^N$ pour un certain entier N. Il s'agit déjà d'une restriction très importante.

Ces observations montrent déjà que si une variété admet une métrique de Kähler-Einstein, alors le signe de sa courbure est imposée par la topologie de X, et plus précisément sa première classe de Chern. Cette situation est très particulière au cas kählerien bien sûr. On retrouve d'ailleurs bien le cas des surfaces (ie courbes complexes compactes) où la formule de Gauss-Bonnet

$$\int_{\Sigma} K dA = 2\pi\chi(\Sigma)$$

montrait bien que le signe de la métrique à courbure constante était imposé par la topologie de la surface.

Rappelons que le cas des courbes est bien compris depuis longtemps. En effet, si la courbe est de genre 0, alors nécessairement il s'agit de $\mathbb{P}^1$ qui possède la métrique de Fubini-Study, à courbure constante positive. En genre 1, ce sont les courbes elliptiques ou tores complexes de dimension 1, ie $\mathbb{C}/\Lambda$ pour un réseau $\Lambda \subset \mathbb{C}$. Alors la métrique plate de $\mathbb{C}$, invariante par translation, descend bien sur le tore en une métrique plate. Enfin, pour les courbes de genre $\geqslant 2$, encore appelées surfaces hyperboliques, il est bien connue qu'elle sont revêtues par le disque unité $\Delta \subset \mathbb{C}$ qui possède la métrique de Poincaré $\frac{|dz|^2}{(1-|z|^2)^2}$ à courbure constante négative, et invariante par tous les automorphismes du disque (c'est conséquence du lemme de Schwarz) ; donc la métrique en question descend bien à la surface hyperbolique en question.

1.1.1.2. La conjecture de Calabi. — Pour résumer la discussion précédente sur l'existence d'une métrique de Kähler-Einstein, trois cas se dégagent :

- La courbure est strictement négative : alors le fibré canonique K_X est ample, et la variété est projective,
- La courbure est nulle : alors K_X est numériquement trivial, ie $c_1(K_X) = 0$ dans $H^{1,1}(X, \mathbb{R})$,
- La courbure est strictement positive : alors K_X^{-1} est ample, et la variété est projective.

Dans le dernier cas, une variété telle que K_X^{-1} est ample est appelée variété de Fano. Comme on peut le voir dans le cas des courbes, il s'agit de variétés plutôt "rigides" et relativement rares. En revanche, le premier cas est plutôt générique ; et de telles variétés s'appelent d'ailleurs variétés de type général. Quant au second, très riche et moins contraignant (pas de projectivité), il est également très lié à de nombreux problèmes venant de la physique théorique.

On peut maintenant formuler un premier problème : soit X une variété d'un des trois types précédents. Admet-elle une métrique de Kähler-Einstein ? Le cas échéant, cette métrique est-elle unique ?

En fait, il existe une formulation plus forte de ce problème généralisant le cas de courbure nulle, qui est la suivante :

***Conjecture 1.1.1* (Calabi).** — *Soit X une variété kählerienne compacte, $[\Omega] \in H^{1,1}(X, \mathbb{R})$ une classe kählerienne, et $\theta \in c_1(K_X^{-1})$ une forme de type $(1,1)$. Existe-t-il* *une métrique kählerienne ω appartenant à la classe de cohomologie $[\Omega]$ telle que*

$$\mathrm{Ric}\,\omega = \theta$$

et le cas échéant, est-elle unique ?

Autrement dit, est-ce que toute $(1,1)$-forme vivant dans la même classe de cohomologie que Ric est elle-même la forme de Ricci d'une métrique kählerienne appartenant à une classe (kählerienne) fixée à l'avance ? Cette question est connue sous le nom de *problème de Calabi* ou encore *conjecture de Calabi* en référence à Eugenio Calabi qui l'a formulée dans les années 50. Il est clair qu'une réponse affirmative à cette conjecture implique l'existence de métriques Ricci plates sur les variétés vérifiant $c_1(K_X) = 0$.

Nous voilà donc avec trois questions : existence de métriques de Kähler-Einstein à courbure négative (resp. positive) sur les variétés canoniquement polarisées (resp. sur les variétés de Fano), et représentabilité de certaines formes comme formes de Ricci.

Le cas de courbure négatif a été résolu par l'affirmative par Aubin et Yau dans les années 70, Yau démontrant par la même occasion la conjecture de Calabi, techniquement bien plus difficile que le premier cas. Nous allons expliquer la preuve de ces résultats dans le prochain paragraphe.

Quant au cas de la courbure positive, des obstructions à l'existence ont été découvertes par Matshushima (réductivité du groupe d'automorphisme) et Futaki (existence d'un invariant lié aux champs de vecteurs) donnant des contre-exemples à l'existence dès la dimension 2 (par exemple $\mathbb{P}^2$ éclaté en un point). Le cas des des surfaces a été complètement traité par Tian, et il restait jusqu'à récemment une conjecture ouverte due à Yau, Tian et Donaldson reliant l'existence de métriques de Kähler-Einstein sur les variétés de Fano à la propriété de K-stabilité de la variété en question. Cette conjecture a été résolue par Chen, Donaldson, Sun [**CDS12a, CDS12b, CDS13**] et Tian [**Tia13**] en utilisant des techniques dont certaines apparaîtront dans cette thèse (métriques KE à singularités coniques).

1.1.2. Equations de Monge-Ampère et résolution de la conjecture. — Une variété kählerienne compacte possède la propriété appelée "lemme du $\partial\bar{\partial}$" qui se résume en disant que deux $(1,1)$-formes $\bar{\partial}$-cohomologues sont en fait $\partial\bar{\partial}$-cohomologues. Ceci permet de ramener l'équation (**KE**) entre formes différentielles à une équation sur le potentiel de la métrique inconnue ω (ie "la" fonction φ qui vérifie $\omega = \omega_0 + i\partial\bar{\partial}\varphi$ pour une métrique kählerienne fixe ω_0 dans la bonne classe de cohomologie). Dans un premier temps, cette équation en φ est donc d'ordre 4, mais une nouvelle utilisation du lemme du $\partial\bar{\partial}$ permet de faire baisser l'ordre de l'équation à deux, et on se ramène à résoudre une équation du type :

$$(\mathbf{MA}) \qquad (\omega_0 + i\partial\bar{\partial}\varphi)^n = e^{-\lambda\varphi + f}\omega_0^n$$

où f est une certaine fonction lisse dépendant de la géométrie de (X, ω_0), qu'on appelle potentiel de Ricci de ω_0. Une telle équation s'appelle *équation de Monge-Ampère*.

Ainsi, trouver des métriques de Kähler-Einstein, ou résoudre le problème de Calabi, revient à résoudre les équations (**MA**) ci-dessus. La stratégie (habituelle) générale pour étudier ces équations s'appelle la méthode de continuité. L'idée est très simple en soi : on veut résoudre une équation (E), qu'on écrit comme la valeur au temps 1 d'un chemin d'équations du même type, notées (E_t). On choisit le chemin de sorte que l'équation (E_0) soit facilement résoluble ; ensuite on considère l'intervalle I des temps $t \in [0,1]$ pour lesquelles l'équation (E_t) a une solution. Le but étant de montrer que $1 \in I$, une manière de le faire est de voir que I vaut l'intervalle $[0,1]$ tout entier. Pour cela, il suffit de montrer que I est ouvert et fermé, et l'on a donc divisé la difficulté en deux parties relativement indépendantes. Le chemin d'équations proposé par Aubin est donnée par

$$(E_t) \qquad (\omega_0 + i\partial\bar{\partial}\varphi)^n = e^{-\lambda\varphi + tf}\omega_0^n$$

En $t = 0$, le choix $\varphi = 0$ donne clairement une solution. Le fait que l'intervalle I soit ouvert résulte du fait que l'opérateur $\varphi \mapsto e^{\lambda\varphi}(\omega_0 + i\partial\bar{\partial}\varphi)^n/\omega_0^n$ est localement inversible lorsque considéré dans les bons espaces fonctionnels (il y a une difficulté supplémentaire lorsque $\lambda > 0$, traitée par Aubin). Sans être complétement évidente, cette partie là n'est pas trop compliquée. La partie plus délicate est la dernière, à savoir montrer que l'intervalle I est fermé.

Il s'agit pour cette partie d'obtenir des estimées a priori. En effet, on dispose d'un chemin de solutions φ_t pour $t \to t_\infty \in]0,1]$, et on désire extraire une sous-suite t_n telle que φ_{t_n} converge uniformément avec toutes ses dérivées vers une fonction φ_{t_∞} solution de (E_{t_∞}). Par le théorème d'Ascoli, il suffit d'obtenir des estimées du type $||\varphi_t||_{\mathscr{C}^k} \leqslant C_k$ pour des constantes C_k ne dépendant que de k (et pas de t).

Il est intéressant de remarquer que pour l'instant, le signe de λ (autrement dit celui de la courbure de la variété) n'importe peu. C'est précisément au niveau des estimées $\mathscr{C}^0$ qu'il va intervenir (et seulement ici). En courbure négative ($\lambda < 0$), une simple application du principe du maximum (disant qu'une fonction $\mathscr{C}^2$ atteignant un maximum local en x_0 a une matrice hessienne négative en ce même point x_0) permet de conclure. En courbure nulle, les choses se compliquent, et c'est d'ailleurs là une des contributions majeures de Yau dans la résolution de la conjecture de Calabi. La preuve consiste à appliquer à la fonction φ_t (dont on sait que son laplacien est uniformément minoré par $-n$) un schéma d'itération de Moser. Sans rentrer dans les détails, l'idée générale est que l'on dispose pour commencer d'une estimée L^2, et on va essayer d'obtenir des estimées L^p bien contrôlées pour p de plus en plus grand, ce qui donnera une estimée L^∞. Pour ce faire, il faut combiner l'hypothèse sur le laplacien avec l'inégalité de Sobolev qui estime la norme $L^{\frac{np}{n-2}}$ de φ_t en fonction des normes L^p de φ_t et de son gradient $\nabla\varphi_t$. Depuis le célèbre papier [**Yau78b**], un mathématicien polonais, S. Kołodziej, a développé dans [**Koł98**] des méthodes de théorie du pluripotentiel pour obtenir des estimées uniformes applicables à toute équation de Monge-Ampère dont le membre de droite est de la forme $f\omega^n$ pour une fonction $f \geqslant 0$ appartenant à $L^p(dV)$ pour un certain $p > 1$. Ces travaux ont eu des répercussions très importantes dans le domaine et ont donné lieu à de nombreuses généralisations en particulier pour les variétés singulières [**EGZ09**].

Enfin, en courbure positive, une telle estimée n'existe pas en général, et la méthode de continuité s'arrête en un $t_\infty < 1$ en général. Ce nombre, parfois appelé "plus grand borne inférieure de Ricci" (en anglais : *greatest lower Ricci bound*) a été beaucoup étudié, en particulier récemment en lien avec son analogue conique, cf [**Don12, Szé11, Li11, LS12, SW12**].

On suppose maintenant que $\lambda \leqslant 0$. Une fois l'estimée $\mathscr{C}^0$ obtenue , il faut monter en régularité (ne serait-ce que pour faire converger une sous-suite). La première étape est d'obtenir une estimée de laplacien, c'est à dire qu'on veut avoir $C^{-1}\omega \leqslant \omega + i\partial\bar{\partial}\varphi_t \leqslant C\omega$ pour une constante $C > 0$ indépendante de t. La méthode est très jolie, et repose sur une inégalité qui compare le laplacien (par rapport à la métrique "inconnue" $\omega + i\partial\bar{\partial}\varphi_t$) de la trace $\mathrm{tr}_\omega(\omega + i\partial\bar{\partial}\varphi_t)$ à certaines quantités géométriques impliquant notamment la courbure de ω : plus précisément, on a :

$$\Delta_{\omega_{\varphi_t}}(\log \mathrm{tr}_\omega \omega_{\varphi_t}) \geqslant \frac{t\Delta f}{\mathrm{tr}_\omega \omega_{\varphi_t}} - B\mathrm{tr}_{\omega_{\varphi_t}}\omega$$

où B ne dépend que d'une borne inférieure de la courbure bisectionnelle de ω. En appliquant à nouveau le principe du maximum à cette inégalité, on arrive (avec un peu de travail) à l'estimée recherchée. Nous aurons l'occasion d'y revenir plusieurs fois dans la suite.

Enfin, un argument similaire (mais plus technique) permet de déduire un contrôle des dérivées d'ordre trois de φ_t, et ensuite l'ellipticité de l'opérateur de Monge-Ampère nous permet de contrôler toutes les dérivées d'ordre supérieure. Notons que l'estimée $\mathscr{C}^3$ n'est plus vraiment utilisée aujourd'hui et a été remplacée avantageusement par la théorie d'Evans-Krylov qui permet de déduire de l'estimée de laplacien une estimée $\mathscr{C}^{2,\alpha}$ ce qui suffit pour utiliser les théorèmes de Schauder et le "bootstrapping".

Ainsi, la recherche de métrique de Kähler-Einstein a été ramenée à l'existence de solutions de certaines équations de Monge-Ampère. La stratégie de résolution de ces équations est la méthode de continuité qui permet de résoudre des équations de plus en plus proches de celle qui nous intéresse, et via l'établissement d'estimées a priori sur les solutions approchées, on est en mesure (ou pas, cf le cas de courbure positive) d'obtenir une solution en passant à la limite. Cette technique ne produit pas de solution explicite, mais elle a l'avantage d'être extrêmement souple, et a pu être très largement généralisée, même déjà dans l'article [**Yau78b**].

Essentiellement toute cette thèse est dévouée à la recherche de métriques de Kähler-Einstein dans des situations plus dégénérées (par exemple l'équation $\mathrm{Ric}\,\omega = \lambda\omega + [D]$ pour un certain diviseur D), ou encore lorsque la variété elle-même est une variété algébrique singulière. Nous verrons comment adapter la stratégie esquissée plus haut à ces nouvelles situations, et dans le cas des variétés singulières, on verra que l'approche variationnelle, très différente, permet d'obtenir des solutions (certes faibles) alors que la méthode "estimées" y échoue.

1.2. Métriques à singularités coniques

1.2.1. Qu'est-ce qu'une métrique conique ?— Commençons par le cas modèle, à savoir celui de $\mathbb{C}$, avec un angle $2\pi\beta$ en 0, pour $\beta \in (0,1)$. On considère deux rayons issus de l'origine écartés d'un angle $2\pi\beta$, et on les recolle. La métrique euclidienne induit alors sur cet espace la métrique

$$g_\beta := dr^2 + \beta^2 r^2 d\theta^2$$

appelée métrique conique d'angle $2\pi\beta$. Alors $\mathbb{C}_\beta := (\mathbb{C}, g_\beta)$ est singulier en l'origine, et il est facile de voir que la métrique riemannienne g_β induite sur $\mathbb{C}^*$ est incomplète. Dans les coordonnées complexes, g_β induit sur $\mathbb{C}^*$ la forme kählerienne

$$\omega_\beta := \frac{idz \wedge d\bar{z}}{|z|^{2(1-\beta)}}$$

On peut ensuite facilement généraliser cette construction à $\mathbb{C}^r$ en choisissant des angles $\beta_1, \ldots, \beta_k$, puis en prenant le produit des espaces riemanniens correspondant. On dit alors que la métrique (produit)

$$\sum_{k=1}^{r} \frac{idz_k \wedge d\bar{z}_k}{|z_k|^{2(1-\beta_j)}} + \sum_{k=r}^{n} idz_k \wedge d\bar{z}_k$$

est à singularités coniques d'angles $2\pi\beta_k$ le long de $[z_k = 0]$, ou encore qu'elle a des singularités coniques le long du diviseur $D = \sum_{k=1}^{r} (1 - \beta_k)[z_k = 0]$.

On aimerait désormais essayer de donner un sens à cette définition de manière intrinsèque pour des variétés. Une définition possible est alors la suivante :

Définition 1.2.1. — Soit X une variété kählerienne de dimension n, et $D = \sum_{k=1}^{r}(1 - \beta_k)D_k$ un diviseur dont le support est à croisements normaux (simples) et tel que $\beta_k \in (0, 1)$ pour tout k. Un courant positif fermé ω de type $(1, 1)$ est dit à singularités coniques le long de D si pour chaque ouvert trivialisant $U \subset X$ où la paire (U, D_U) est isomorphe à $(\mathbb{C}^n, (1 - \beta_{j_1})[z_1 = 0] + \cdots + (1 - \beta_{j_s})[z_s = 0])$, il existe une constante $C_U > 0$ telle que :

$$C^{-1}\omega \leqslant \sum_{k=1}^{s} \frac{idz_k \wedge d\bar{z}_k}{|z_k|^{2(1-\beta_{j_k})}} + \sum_{k=r}^{n} idz_k \wedge d\bar{z}_k \leqslant C\omega$$

Il est clair que cette définition est bien cohérente au sens où il suffit de la vérifier pour un seul recouvrement de X par des ouverts trivialisants. De plus, si X est compacte, alors on peut aussi bien demander que la constante C_U soit universelle (ie indépendante de U).

Maintenant que l'on sait ce qu'est une métrique conique se pose la question fatidique : à quoi ça sert ? Ou encore, pourquoi regarder ce type de métriques ? On peut donner plusieurs justifications à l'étude de telles métriques, certaines (d'ailleurs cruciales) venant seulement a posteriori.

Dès les années 90, Troyanov [**Tro91**] et McOwen [**McO93**] s'intéressent à la question de l'existence de métriques coniques sur des surfaces de Riemann épointées dont la courbure est prescrite. Naturellement, il existe des obstructions topologiques (provenant par exemple de la formule de Gauss-Bonnet), mais ce sont essentiellement les seules, ce que l'on retrouvera en partie en plus grande dimension dans [**CGP11**]. Ceci nous permettra d'ailleurs d'obtenir des applications quant à l'existence de certains tenseurs holomorphes dont les zéros et les pôles près d'un diviseur snc sont prescrits.

Du point de vue de la géométrie riemannienne, les cones métriques et en particuliers les $\mathbb{C}_\beta \times \mathbb{C}^{n-1}$ sont des objets très importants, car ils apparaissent naturellement comme les cones tangents de certaines limites (au sens de Gromov-Hausdorff) de variétés riemanniennes ; c'est un point crucial de l'approche de Chen-Donaldson-Sun pour la résolution de la conjecture de Yau-Tian-Donaldson, cf [**CDS12b**].

Enfin, ce qui a vraisemblablement rendu ces questions très populaires est l'article [**Don12**] où Donaldson explique qu'une approche possible pour la fameuse conjecture est de faire une méthode de continuité où l'on considérera des métriques de Kähler-Einstein *coniques* au sens où elles vérifieront $\mathrm{Ric}\,\omega_\beta = \beta\omega + (1-\beta)[D]$, et de voir si on peut faire tendre l'angle vers 1 auquel cas on devrait obtenir des métriques de Kähler-Einstein au sens usuel. L'introduction de singularités dans ce programme pouvait au premier abord sembler compliquer l'approche, mais en réalité ce programme a été conduit à son terme par Chen-Donaldson-Sun, et Tian.

1.2.2. Métriques de Kähler-Einstein coniques. — Au vu de la discussion précédente, les métriques à singularités coniques présentent un intérêt tout particulier lorsque qu'elles sont de Kähler-Einstein (en un sens à préciser). Après avoir défini précisément ce que signifie la courbure de Ricci de telles métriques singulières, nous présenterons dans le paragraphe suivant les résultats obtenus en collaboration avec Frédéric Campana et Mihai Păun dans [**CGP11**][(1)].

Le contexte est le suivant : X est une variété kählerienne compacte, et $D = \sum(1-\beta_i)D_i$ est un diviseur à support snc tel que $\beta_i \in (0,1)$. Le but est de trouver sous quelles conditions $X \setminus D$ a une métrique de Kähler-Einstein (lisse donc) qui soit à singularités coniques le long de D au sens où la métrique est quasi-isométrique à la métrique conique standard dans les coordonnées près des points de D, cf définition 1.2.1.

Il est évident que ce n'est pas en résolvant $\mathrm{Ric}\,\omega = \lambda\omega$ sur $X\setminus D$ qu'on va pouvoir retrouver une métrique conique avec les bons angles (il y aurait a priori beaucoup trop de solutions), et il est important d'essayer de formuler le problème globalement sur X, par exemple en termes d'équations de Monge-Ampère. Essayons de comprendre ce qu'il se passe localement déjà. Si ω est une métrique kählerienne sur $\mathbb{C}^*$ qui est à singularités coniques d'angle $2\pi\beta$ en 0, alors il est clair que ω se prolonge en un courant positif fermé sur $\mathbb{C}$ par exemple par le théorème de Skoda El-Mir car ω est bien de masse finie près de 0. On écrit

$$\omega = e^f \frac{idz \wedge d\bar{z}}{|z|^{2(1-\beta)}}$$

pour une fonction f bornée. On voit alors que $\log(\omega/idz \wedge d\bar{z})$ est bien localement intégrable, et on peut alors donner un sens à $\mathrm{Ric}\,\omega := -dd^c \log(\omega/idz \wedge d\bar{z})$ ce que l'on appelle par définition la courbure de Ricci de ω au sens des courants. Dans notre cas, on a donc $\mathrm{Ric}\,\omega = (1-\beta)[0] - dd^c f$. Par exemple, la métrique référence ω_β introduite plus haut est Ricci-plate au sens conique : $\mathrm{Ric}\,\omega_\beta = (1-\beta)[0]$. La partie "conique" de la métrique doit donc se refléter au niveau de sa courbure de Ricci, faisant apparaître le courant d'intégration sur le diviseur avec les bons coefficients. Ces observations suggèrent la définition suivante :

Définition 1.2.2. — Soit (X, D) comme précédemment. Une métrique kählerienne ω sur $X \setminus \mathrm{Supp}(D)$, de masse finie est appelée métrique Kähler-Einstein pour (X, D) si $\log(\omega^n/dV)$ est localement intégrable, et s'il existe un réel λ tel que

$$(\mathbf{KE_c}) \qquad \mathrm{Ric}\,\omega = \lambda\omega + [D]$$

au sens des courants.

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(1) Les résultats de cet article peuvent être considérés comme préliminaires à notre thèse, et ont servi de point de départ pour les diverses généralisations obtenues dans ce travail.

Avant d'aller plus loin, faisons trois observations :

La première est de dire que comme dans le cas standard, il y a des obstructions cohomologiques à l'existence de telles métriques ; pour que ($\mathbf{KE_c}$) puisse avoir une solution, il faut nécessairement que $c_1(K_X + D)$ puisse contenir $-\lambda\omega$. Donc en courbure négative (resp. positive, nulle), $K_X + D$ doit être ample (resp. anti-ample, numériquement trivial).

La seconde est que vue la définition choisie de la courbure de Ricci, on peut encore formuler l'équation Kähler-Einstein conique ($\mathbf{KE_c}$) en termes d'équation de Monge-Ampère sous la forme

$$(\mathbf{MA_c}) \qquad (\omega_0 + i\partial\bar{\partial}\varphi)^n = \frac{e^{-\lambda\varphi+f}}{\prod |s_i|^{2(1-\beta_i)}}\,\omega_0^n$$

pour des sections s_i définissant D_i, des métriques hermitiennes lisses $|\cdot|_i$ sur $\mathscr{O}_X(D_i)$, et une fonction $f \in \mathscr{C}^\infty(X)$. Il apparaît cependant une difficulté car on sait a priori que le potentiel φ ne peut pas être lisse. Donc il faut définir la classe de fonctions dans laquelle donner un sens et résoudre l'équation de Monge-Ampère. En réalité, on peut voir (par exemple grâce au théorème de Kołodziej [**Koł98**], mais on peut s'en sortir autrement également) qu'il suffit de travailler avec des fonctions ω-psh qui sont *bornées*, auquel cas la définition de l'opérateur de Monge-Ampère est bien comprise grâce aux travaux fondateurs de Bedford et Taylor [**BT82**].

La dernière remarque, par laquelle on aurait peut-être même dû commencer, est qu'il n'est pas clair du tout qu'une métrique Kähler-Einstein au sens conique soit bien à singularités coniques le long de D ! Or, c'est exactement ce que l'on veut. Car en dimension plus grande que 1, imposer que la forme volume ω^n soit à constantes multiplicatives près la même qu'une forme volume conique ne dit a priori pas grand chose sur le comportement de la métrique elle-même près du diviseur. Du point de vue de l'algèbre linéaire, cela revient à passer de l'information qu'une matrice hermitienne (à coefficients dans $\mathscr{C}^\infty\left(\mathbb{B}_{\mathbb{C}^n}(1) \setminus \{z_1 = \ldots = z_k = 0\}\right)$) ayant un déterminant contrôlé près du diviseur (disons dans $[C^{-1}, C]$) à l'information que toutes ses valeurs propres sont dans un tel intervalle.

Ainsi, on sait qu'une métrique conique et Kähler-Einstein sur $X\setminus\mathrm{Supp}(D)$ doit vérifier une équation du type ($\mathbf{MA_c}$), mais il n'est pour l'instant pas clair que les éventuelles solutions de cette équation sont bien à singularités coniques. D'ailleurs (et tant pis pour le suspense), ce n'est pas encore connu en toute généralité à l'heure actuelle.

1.2.3. Résultats de [CGP11]. — L'énoncé du théorème principal de [**CGP11**] a l'avantage d'être très simple à comprendre : il répond à la question soulevée ci-dessus, sous l'hypothèse que les angles des cones ne sont pas trop grands :

Théorème 1.2.3. — *Soient X une variété kählerienne compacte, $D = \sum(1-\beta_i)D_i$ un diviseur effectif à support snc, $\mu \in \mathbb{R}$, et supposons de plus que $\beta_i \leqslant 1/2$ pour tout i. Alors toute solution $\omega = \omega_0 + i\partial\bar{\partial}\varphi$ de l'équation*

$$(\omega_0 + i\partial\bar{\partial}\varphi)^n = \frac{e^{\mu\varphi+f}}{\prod |s_i|^{2(1-\beta_i)}}\,\omega_0^n$$

est à singularités coniques le long de D.

Par conséquent, toute métrique de Kähler-Einstein pour la paire (X, D) est bien à singularités coniques le long de D si les coefficients de ce dernier sont plus grands que $1/2$:

Corollaire 1.2.4. — *Sous les hypothèses du théorème précédent, alors :*

- *Si K_X+D est ample, alors il existe une unique métrique de Kähler-Einstein à courbure négative qui soit à singularités coniques le long de D.*
- *Si K_X+D est trivial, alors il existe dans chaque classe kählerienne une unique métrique Ricci-plate qui soit à singularités coniques le long de D.*
- *Si $-(K_X+D)$ est ample, alors toute métrique de Kähler-Einstein (au sens faible) à courbure positive est à singularités coniques le long de D.*

Il y a essentiellement deux stratégies naturelles pour essayer de prouver un résultat comme le Théorème 1.2.3 si on a en tête le cas standard où $D=0$. L'idée générale est d'approcher l'équation qui nous intéresse par des équations proches qu'on sait résoudre. Déjà dans [**Yau78b**], Yau s'intéresse à des équations au membre de droite ayant zéros et pôles. Pour montrer l'existence de solutions, il régularise ces fonctions (transformant $|s|^{2a}$ en $(|s|^2+\varepsilon^2)^a$) puis il obtient des estimées loin du lieu singulier ce qui lui permet de dire que, sous certaines hypothèses sur les pôles, il existe une solution lisse loin des singularités de l'équation. Nous voulons faire exactement la même chose, mais en conservant une information précise près des singularités, ce qui n'est pas complètement déraisonnable étant donné que notre membre de droite est relativement sympathique. Comme nous le verrons, cette stratégie requiert de connaître précisément la géométrie des métriques coniques régularisée, et en particulier leur courbure. Mentionnons au passage l'article [**CDS12a**], ou la même stratégie est employée (mais pour obtenir des estimations plus faibles qui suffisent à obtenir les renseignements nécessaires pour [**CDS12b, CDS13**]).

Mais on peut également vouloir travailler directement dans les espaces de métriques coniques, y développer des theorèmes d'inversion locale, des estimées de laplacien, et également les estimées $\mathscr{C}^{2,\alpha}$ d'Evans-Krylov. C'est l'approche développée par Brendle [**Bre11**] et [**JMR11**] qui fournit des résultats à la fois plus et moins généraux. En effet, les espaces fonctionnels en question sont des espaces de potentiels de métriques coniques, mais où plus d'information est connue près du diviseur. Mais pour l'instant, seulement le cas d'une composante peut être traité ainsi, et il peut également y avoir des restrictions sur les coefficients.

Revenons à la stratégie de régularisation. Un de ses grands avantages est sa souplesse, car comme nous le verrons, nous étendrons cette méthode pour résoudre le cas où le diviseur a des coefficients éventuellement égaux à 1 (cf section 1.3 ou [**Gue12b**]), ainsi que le cas où la paire (X, D) est seulement une paire klt, donc en particulier X peut être singulière, et le diviseur D est absolument quelconque (ou presque), cf section 1.4 ou [**Gue12a**]. Bien sûr, les conclusions dans ce cadre très général là sont plus faibles, ce à quoi on pouvait logiquement s'attendre.

On ne va pas rentrer dans les détails de la preuve du Théorème 1.2.3, mais en donner seulement les grandes lignes pour comprendre également d'où vient l'hypothèse sur les coefficients. Comme nous l'avons expliqué, on commence par résoudre l'équation approchée

$$(\omega_0 + i\partial\bar{\partial}\varphi_\varepsilon)^n = \frac{e^{\mu\varphi_\varepsilon + f}}{\prod(|s_i|^2 + \varepsilon^2)^{1-\beta_i}}\,\omega_0^n$$

Il y a une difficulté si μ est négatif, mais faisons comme si de rien n'était. La première chose à faire est d'introduire un métrique $\omega_\varepsilon = \omega_0 + dd^c\psi_\varepsilon$ qui converge vers une métrique à singularités coniques. Ensuite, on change de potentiel en posant $u_\varepsilon := \varphi_\varepsilon - \psi_\varepsilon$, et on réécrit l'équation comme

$$(\omega_\varepsilon + i\partial\bar{\partial}u_\varepsilon)^n = e^{\mu u_\varepsilon + F_\varepsilon}\,\omega_\varepsilon^n$$

Il faut obtenir des estimées sur les dérivées (par rapport à la métrique conique approchée) de φ_ε qui soient indépendantes de ε. La bonne chose est que l'on a fait disparaître les pôles ainsi, et les fonctions F_ε sont mêmes uniformément bornés.

Alors, il nous suffit maintenant, comme on l'a déjà expliqué d'avoir sur u_ε des estimées $\mathscr{C}^0$ et des estimées de laplacien (par rapport à ω_ε bien sûr). Les premières s'obtiennent facilement soit par le principe du maximum, soit par l'estimée de Kołodziej, car le membre de droite de l'équation de Monge-Ampère initiale est bien dans L^p pour un certain $p > 1$ car tous les β_i sont strictement positifs. Vient ensuite l'estimée de laplacien, qui constitue le coeur du travail.

En utilisant l'inégalité de laplacien rappelée dans la première partie, on voit que tout revient à minorer uniformément la courbure bisectionnelle de ω_ε, ainsi que le laplacien $\Delta_{\omega_\varepsilon} F_\varepsilon$; où l'on rappelle que F_ε est le logarithme des rapports entre la forme volume de la métrique conique approchée, ω_ε^n, et de $dV/\prod(|s_i|^2 + \varepsilon^2)^{1-\beta_i}$.

A partir de la, il est difficile de "raconter" la preuve avec des mots ; il s'agit essentiellement de choisir astucieusement les coordonnées locales, expliciter tous les termes impliqués dans la courbure (et le laplacien en question), et voir que *sous l'hypothèse sur les coefficients*, à savoir $\beta_i \leqslant 1/2$, on obtient bien une minoration des termes. On peut noter que la courbure bisectionnelle de ω_ε n'est *pas* uniformément majorée, même sous l'hypothèse du théorème, et ce dès la dimension 1.

Signalons maintenant que la motivation du Théorème 1.2.3 venait d'une question posée par F. Campana, concernant l'annulation des tenseurs holomorphes orbifoldes. Nous donnerons la définition précise de ces objets dans le chapitre 2, mais pour l'instant voyons ces objets comme des sections au-dessus de $X \setminus \mathrm{Supp}(D)$ du fibré holomorphe holomorphe $T^r_s X := (\bigotimes^r T_X) \otimes (\bigotimes^s T^*_X)$ qui sont *bornées* pour une métrique conique (ou disons par rapport à la métrique induite sur ce fibré par une métrique à singularités coniques le long de D). Ainsi, la partie covariante (T^r) doit s'annuler à un certain ordre le long de D (croissant avec r) alors que la partie contravariante (T_s) est autorisée à avoir des pôles controlés par s et les coefficients de D. On notera l'espace des tenseurs holomorphes orbifoldes par $H^0(X, T^r_s(X|D))$.

En utilisant la formule de Bochner, des fonctions tronquantes appropriée à la géométrie conique et le Corollaire 1.2.4, on arrive à généraliser un théorème de Kobayashi :

Théorème 1.2.5. — *Sous les hypothèses du Théorème 1.2.3*[(2)]*, alors :*

[(2)] Si ce théorème était généralisé pour des angles $\beta_i \in (0,1)$ quelconques, alors le Théorème 1.2.5 serait également valable sans restriction sur les angles.

- *Si* $K_X + D$ *est ample, alors* $H^0(X, T_s^r(X|D)) = 0$ *pour tout* $r \geqslant s+1$.
- *Si* $-(K_X + D)$ *est ample, alors* $H^0(X, T_s(X|D)) = 0$ *pour tout* $s \geqslant 1$.
- *Si* $c_1(K_X + D)$ *contient un représentant lisse semi-positif (resp. semi-negatif), alors les sections holomorphes de* $T^r(X|D)$ *(resp.* $T_s(X|D)$*) sont parallèles.*

1.3. Métriques à singularités mixtes Poincaré et coniques

1.3.1. La géométrie des métriques de Poincaré. — On a vu dans la section précédente une famille naturelle de géométries, paramétrées par un angle de cone. Du point de vue de la géométrie riemannienne pourtant, ces métriques ne se comportent pas si bien que ça (elles sont incomplètes, et surtout ont un tenseur de courbure non borné près du sommet du cône) ce qui les rend délicates à manipuler pour les rendre Kähler-Einstein, comme on l'a vu plus haut.

Mais en fait, il existe une métrique sur le disque épointé bien plus naturelle que les métriques coniques ; il s'agit de la métrique de Poincaré :

$$\frac{|dz|^2}{|z|^2 \log^2 |z|^2}$$

C'est une métrique complète à courbure constante négative ; c'est simplement la métrique hyperbolique sur le disque unité descendue via le revêtement universel. On peut alors exporter ce modèle comme dans le cas conique à la situation d'une variété kählerienne compacte munie d'un diviseur snc :

Définition 1.3.1. — Soit X une variété kählerienne de dimension n, et $D = \sum_{k=1}^r D_k$ un diviseur à croisements normaux (simples). Un courant positif fermé ω de type $(1,1)$ est dit à singularités Poincaré le long de D si pour chaque ouvert trivialisant $U \subset X$ où la paire (U, D_U) est isomorphe à $(\mathbb{D}^n, [z_1 = 0] + \cdots + [z_s = 0])$, il existe une constante $C_U > 0$ telle que :

$$C^{-1}\omega \leqslant \sum_{k=1}^{s} \frac{i dz_k \wedge d\bar{z}_k}{|z_k|^2 \log^2 |z_k|^2} + \sum_{k=r}^{n} i dz_k \wedge d\bar{z}_k \leqslant C\omega$$

Il est alors clair qu'une métrique de type Poincaré induit une métrique complète sur $X \setminus D$.

Insistons un peu sur la naturalité de telles métriques. Depuis Aubin et Yau, on sait construire sur une variété canoniquement polarisée une (unique) métrique de Kähler-Einstein. Très vite s'est posée la question de savoir à quels types de variétés pouvait s'étendre une telle construction. Tout d'abord, Cheng-Yau [**CY80**] puis Mok-Yau [**MY83**] se sont intéressés au cas des domaines d'holomorphie dans $\mathbb{C}^n$, pour lesquels ils ont prouvé l'existence d'une (unique) métrique de Kähler-Einstein à courbure négative ; en fait ils prouvent que l'existence d'une telle métrique sur un domaine $\Omega \subset \mathbb{C}^n$ est *équivalente* à ce que Ω soit un domaine d'holomorphie. Notons que l'unicité d'une telle métrique est une conséquence directe du lemme de Yau-Schwarz [**Yau78a**], qui lui se déduit du principe du maximum généralisé aux variétés complètes par Yau [**Yau75**].

Dans la continuité de ces travaux, R. Kobayashi [**Kob84**] et Tian-Yau [**TY87**] ont ensuite étudié la question de l'existence de métriques de Kähler-Einstein complètes sur des variétés quasi-projectives de type $X \setminus D$ où X est kählerienne compacte, et D est un diviseur snc. Ils ont prouvé que dès lors que $K_X + D$ est ample (condition analogue au cas compact), alors il existe sur $X \setminus D$ une unique métrique KE à courbure négative (disons -1), et que celle-ci est de type Poincaré en un sens assez fort ; en particulier elle induit une géométrie bornée. Ainsi, la géométrie Poincaré apparaît très naturellement dans le cadre des métriques KE complètes à courbure négative sur des variétés quasi-projectives.

Ce qui rend ces métriques très maniables, c'est essentiellement leur propriété d'être à *géométrie bornée*. Cela signifie qu'il existe au voisinage des points de D des quasi-coordonnées (ie une famille de difféomorphismes locaux d'une boule euclidienne fixe vers les voisinages en questions, mais pas injectifs en général) dans lesquels la métrique est équivalente à la métrique euclidienne, et telle que tous ses coefficients ont des dérivées uniformément bornées ; en particulier, son tenseur de courbure est bien borné. Il est alors possible de reproduire la méthode de continuité habituelle en utilisant ces quasi-coordonnées, et quant aux estimées a priori, elles découlent essentiellement des estimées usuelles ainsi que du principe du maximum de Yau. C'est un des endroits cruciaux où la complétude est utilisée.

1.3.2. Singularités mixtes Poincaré et coniques. — Bien que les géométries coniques et de Poincaré soient fondamentalement très différentes, elles sont en fait intimement reliées au sens où la géométrie de Poincaré est en quelque sorte la limite des géométries coniques lorsque l'angle $2\pi\beta$ tend vers 0 ; informellement, le cône va se déformer en un *cusp*. On peut donner à cette assertion une justification un peu moins heuristique également : considérons sur $\mathbb{D}^*$ la métrique KE suivante, conique d'angle $2\pi\beta$ en 0 :

$$\omega_\beta = \frac{\beta^2 i dz \wedge d\bar{z}}{|z|^{2(1-\beta)}(1-|z|^{2\beta})^2}$$

Alors un calcul facile montre que lorsque $\beta \to 0$, $\omega_\beta \to \omega_P = \frac{idz\wedge d\bar{z}}{|z|^2 \log^2 |z|^2}$, la métrique de Poincaré du disque épointé.

Question. Au vu des résultats ci-dessus et de ceux de la section 1.2, on peut se demander si l'on peut "mélanger" ces deux géométries dans le cadre Kähler-Einstein. Plus précisément, étant donnée une variété kählerienne compacte X, un diviseur $D = \sum_{j=1}^r (1-\beta_j)D_j + \sum_{k=r+1}^s D_k$ à support snc, et tel que $\beta_j \in (0,1)$ pour tout j, existe-t-il une métrique de Kähler-Einstein à courbure négative sur $X \setminus \mathrm{Supp}(D)$ ayant des singularités coniques le long de $D_{klt} := \sum_{j=1}^r (1-\beta_j)D_j$ et Poincaré le long de $D_{lc} := \sum_{k=r+1}^s D_k$, au sens où la métrique est quasi-isométrique au produit des métriques modèles sur les disques épointés ?

Par souci de complétude, donnons la définition de singularités mixtes Poincaré et coniques, même si elle est très naturelle :

Définition 1.3.2. — Soit X une variété kählerienne de dimension n, et $D = \sum_{j=1}^r (1-\beta_j)D_j + \sum_{k=r+1}^s D_k$ un diviseur dont le support est à croisements normaux (simples) et tel que $\beta_j \in (0,1)$ pour tout j. Un courant positif fermé ω de type $(1,1)$ est dit à singularités mixtes Poincaré et coniques le long de D si pour chaque ouvert trivialisant $U \subset X$ où la paire (U, D_U) est isomorphe à $(\mathbb{C}^n, (1-\beta_{j_1})[z_1 = 0] + \cdots + (1-\beta_{j_p})[z_p = 0] + [z_{p+1} = 0] + \cdots + [z_q = 0])$, il existe une constante $C = C_U > 0$ telle que :

$$C^{-1}\omega \leqslant \sum_{i=1}^{p} \frac{idz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\beta_{j_i})}} + \sum_{k=p+1}^{q} \frac{idz_k \wedge d\bar{z}_k}{|z_k|^2 \log^2 |z_k|^2} + \sum_{l>q} idz_l \wedge d\bar{z}_l \leqslant C\omega$$

Le but de cette partie est de répondre à la question posée ci-dessus. Tout d'abord, voyons ce que l'existence d'une telle métrique impliquerait.

Pour commencer, et comme on pouvait s'en douter, une condition nécessaire est que $K_X + D$ soit ample. En effet, on peut voir qu'une telle métrique ω sur $X \setminus D$ s'étend automatiquement en un courant dans $c_1(K_X + D)$, dominant une forme de Kähler et n'ayant pas de nombre de Lelong. Ainsi par le théorème de régularisation de Demailly, la classe $c_1(K_X + D)$ est nécessairement ample.

Ensuite, on remarque qu'une métrique KE ω sur $X \setminus D$, à courbure de Ricci égale à -1 et ayant des singularités mixtes Poincaré et coniques le long de D va automatiquement induire une métrique KE pour la paire (X, D) au sens de la définition 1.2.2 : ω s'étend en un courant sur X dont on peut définir globalement la courbure de Ricci, qui vaut alors

$$(\mathbf{KE_{pc}}) \qquad \operatorname{Ric}\omega = -\omega + [D]$$

En termes d'équations de Monge-Ampère, si l'on fixe une métrique kählerienne $\omega_0 \in c_1(K_X + D)$, une solution $\omega = \omega_0 + dd^c\varphi$ à l'équation ci-dessus doit vérifier l'équation suivante globale sur X, au moins de manière formelle :

$$(\mathbf{MA_{pc}}) \qquad (\omega_0 + dd^c\varphi)^n = \frac{e^{\varphi+f}\omega_0^n}{\prod_{j=1}^{r} |s_j|^{2(1-\beta_j)} \prod_{k=r+1}^{s} |s_k|^2}$$

pour une certaine fonction $f \in \mathscr{C}^\infty(X)$.

A ce stade, il y a deux options. La première consiste à essayer de résoudre l'équation $(\mathbf{MA_{pc}})$ sur X, et voir si l'on peut montrer que la solution a bien les singularités espérées, par exemple en utilisant une méthode de régularisation comme dans le cas conique. La difficulté majeure est que un éventuel potentiel solution φ ne peut pas être borné, ce qui complique énormément la tâche pour résoudre l'équation et analyser l'éventuelle solution. En particulier, il faut également donner un sens au Monge-Ampère d'un tel courant, ce qui n'est pas évident a priori.

La deuxième option, assez orthogonale à la première, serait d'oublier l'aspect global de la question, comme c'est le cas dans l'approche de [**Kob84, TY87**] ou encore [**Bre11, JMR11**] dans la situation conique. Il faudrait alors introduire les bons espaces fonctionnels encodant les singularités Poincaré et coniques, puis voir si l'on peut faire marcher la méthode de continuité dans ces espaces là. Mais en fait, déjà le cas conique à plusieurs composantes est ouvert à l'heure actuelle (par ces méthodes), donc il est peu vraisemblable d'arriver à traiter notre problème à moindre coût en utilisant cette approche. Par exemple, obtenir l'unicité d'une telle métrique semble loin d'être évident au vu des techniques très différentes utilisées dans le cas Poincaré (complétude) et conique (argument de perturbation, cf [**Jef00**]).

Revenons donc à la première option. Rappelons donc qu'une des différences fondamentales avec le cas conique, c'est que le potentiel φ n'est pas borné le long de D_{lc} (c'est déjà le cas dans la situation purement Poincaré où le potentiel se comporte en $-\log(\log^2|z|^2)$ essentiellement). Ainsi, comment doit-on définir le produit $(\omega_0+dd^c\varphi)^n$ apparaissant dans l'équation ($\mathbf{MA_{pc}}$) ? La réponse se trouve dans les travaux de Guedj et Zeriahi [**GZ05, GZ07**] où est développée la notion d'opérateur de Monge-Ampère non-pluripolaire. En bref, étant donnée une fonction ω_0-psh φ, on peut lui associer une mesure notée $\mathrm{MA}(\varphi)$ ou encore $(\omega_0+dd^c\varphi)^n$ qui ne charge pas les ensembles pluripolaires, et qui coïncide bien avec l'opérateur de Monge-Ampère habituel si φ est bornée par exemple. De plus, la masse totale de $\mathrm{MA}(\varphi)$ est toujours inférieure ou égale à $\int_X \omega_0^n$. Alors, la classe naturelle de fonctions à considérer est la classe $\mathcal{E}(X,\omega)$ des fonctions ω_0-psh de masse de Monge-Ampère totale (ie égale à $\int_X \omega_0^n$). Pour de nombreuses raisons, il s'agit du bon espace fonctionnel dans lequel travailler pour nos types de problèmes. Mais comment s'assurer que notre équation a bien une solution dans $\mathcal{E}(X,\omega)$?

C'est là un des points clés, bien que pas forcément très compliqué : il nous faut voir que les solutions obtenues par Kobayashi et Tian-Yau (donc dans le cas purement Poincaré) sont bien d'énergie finie. Nous renvoyons au Lemme 2.2.3 pour les détails, qui passent par un calcul (local) de capacité d'ensembles de sous-niveaux. Une fois cette remarque faite, on peut alors mimer les étapes de [**CGP11**], et résoudre pour chaque $\varepsilon>0$ l'équation de Monge-Ampère

$$(\omega_0+dd^c\varphi_\varepsilon)^n=\frac{e^{\varphi_\varepsilon+f}\omega_0^n}{\prod_{j=1}^r(|s_j|^2+\varepsilon^2)^{1-\beta_j}\prod_{k=r+1}^s|s_k|^2}$$

que l'on peut également réécrire

$$(\omega_P+dd^cu_\varepsilon)^n=\frac{e^{u_\varepsilon+f}\omega_0^n}{\prod_{j=1}^r(|s_j|^2+\varepsilon^2)^{1-\beta_j}\prod_{k=r+1}^s|s_k|^2\log^2|s_k|^2}$$

pour $\omega_P:=\omega_0-\sum_k\log\log^2|s_k|^2$ une métrique Poincaré, et $u_\varepsilon:=\varphi_\varepsilon+\sum_k\log\log^2|s_k|^2$ le potentiel renormalisé. La solution obtenue aura bien des singularités Poincaré le long de D_{lc} par l'observation précédente (en particulier u_ε est bornée), et est censée acquérir des singularités coniques le long de D_{klt} à la fin du processus quand $\varepsilon\to 0$.

Pour prouver cela, on va établir comme dans le cas conique des estimées pour φ_ε par rapport à une métrique référence de la forme $\omega_\varepsilon:=\omega_P+dd^c\psi_\varepsilon$ où ψ_ε est le potentiel conique (par rapport à D_{klt}) régularisé comme dans la section précédente. L'équation devient alors

$$(\omega_\varepsilon+dd^cu'_\varepsilon)^n=e^{u'_\varepsilon+F_\varepsilon}\omega_\varepsilon^n$$

et il reste alors à vérifier que la courbure bisectionnelle de ω_ε est bien uniformément minorée indépendamment de ε (on aura donc besoin de l'hypothèse $\beta_j\leqslant 1/2$ à nouveau), puis que le rapport des volumes F_ε a également un laplacien minoré uniformément. Les calculs sont assez lourds, mais il se trouve que les deux géométries conique et Poincaré n'interfèrent qu'assez peu ici, et donc la situation ressemble presque à une situation produit. Finalement, le théorème est le suivant :

Théorème 1.3.3 ([**Gue12b**]). — *Soit X une variété kählerienne compacte, $D=\sum_{j=1}^r(1-\beta_j)D_j+\sum_{k=r+1}^s D_k$ à support snc, tel que $\beta_j\in(0,1/2]$ pour tout j. Si K_X+D*

est ample, alors il existe une unique métrique de Kähler-Einstein ω_{KE} *vérifiant*

$$\operatorname{Ric}\omega_{\mathrm{KE}} = -\omega_{\mathrm{KE}} + [D]$$

et de plus ω_{KE} *est à singularités mixtes Poincaré et coniques le long de* D*.*

On note la dissymétrie avec l'énoncé dans le cas conique, où la courbure pouvait avoir n'importe quel signe. La raison est que la géométrie de Poincaré est fondamentalement à courbure négative. Il est facile d'exclure la courbure positive par le théorème de Bonnet-Myers, et ensuite un argument élémentaire (cf remarque 2.1.3) montre que le cas Ricci-plat n'est pas non plus possible.

Pour finir cette partie, mentionnons une application à l'annulation des tenseurs holomorphes orbifoldes, comme dans la section précédente :

Corollaire 1.3.4. — *Soit* (X, D) *une paire comme précédemment, telle que* $K_X + D$ *est ample. Alors tout tenseur holomorphe de type* (r, s) *avec* $r \geqslant s + 1$ *est nul :*

$$H^0(X, T_s^r(X|D)) = 0.$$

A nouveau, nous n'avons pas donné la définition des tenseurs holomorphes, référant à [**Gue12b**] pour plus de détails. Il est à noter cependant qu'une difficulté supplémentaire apparaît par rapport au cas conique, à savoir que les tenseurs holomorphes orbifoldes pour (X, D) ne sont *plus* les tenseurs holomorphes sur $X \setminus \mathrm{Supp}(D)$ bornés par rapport à une métrique de type Poincaré-conique. Ainsi, pour utiliser la métrique Kähler-Einstein, il faudra comprendre dans la formule de Bochner un terme supplémentaire (qu'on peut interpréter comme une courbure) et voir qu'il ne joue pas (trop) en notre défaveur.

1.4. Métriques de Kähler-Einstein sur les variétés singulières

1.4.1. Un peu de singularités. — Avant de rentrer dans le vif du sujet et d'expliquer pourquoi l'on va essayer de transporter la théorie des métriques de Kähler-Einstein dans le cadre des variétés algébriques singulières, il est raisonnable de redonner quelques définitions concernant les singularités auxquelles on va se restreindre.

Dans essentiellement toute la suite, X va désigner une variété complexe projective normale. En particulier, le lieu des singularités X_{sing} de X a pour codimension au moins 2. On sait depuis Hironaka que X admet une résolution des singularités, c'est-à-dire qu'il existe une variété lisse X' et un morphisme birationnel surjectif $\pi : X' \to X$. De plus, on peut supposer que π est un isomorphisme au dessus de X_{reg} (lieu régulier de X), et plus précisément que π est une succession d'éclatements de centres lisses (inclus dans X_{sing} donc).

Pour une variété X normale, il existe plusieurs moyens (non équivalents) de définir un faisceau canonique généralisant le fibré canonique habituel. Le premier consiste à considérer la puissance extérieure maximale du faisceau des différentielles de Kähler, mais en général on obtient un faisceau non réflexif, objet auquel on préférerait si possible ne pas avoir à faire. La deuxième solution, qui est celle adoptée consiste à définir tout d'abord K_X sur X_{reg} ; on obtient donc un faisceau localement libre de rang un sur un (gros) ouvert, et on l'étend à X tout entier par saturation. Cela revient à choisir un diviseur de Weil D_0 représentant $K_{X|X_{\text{reg}}}$, puis considérer $D := \overline{D}_0$ l'adhérence de D_0, ce qui donne un diviseur de Weil, ou encore un faisceau réflexif de rang un (plus précisément une classe d'équivalence). On notera toujours K_X le diviseur de Weil obtenu, et il est à noter qu'en général, ce diviseur ne correspond pas à un faisceau localement libre (de rang un), comme par exemple pour $\mathbb{C}^3/\{\pm 1\}$.

Du point de vue de la géométrie birationnelle, une classe importante de variété se dégage ; il s'agit des variétés X pour lesquelles le diviseur de Weil K_X, ou même seulement un de ses multiples mK_X est un diviseur de Cartier, autrement dit qu'il provient d'un fibré en droites. On dit que ces variétés sont $\mathbb{Q}$-Gorenstein. On peut alors tirer en arrière K_X (ou mK_X) par $\pi : X' \to X$, et le comparer à $K_{X'}$, ce qui va nous fournir des nombres rationnels appelés discrépances, et qui permettront alors de définir des classes de singularités. Plus précisément, on écrit

$$K_{X'} = \pi^* K_X + \sum_i a_i E_i$$

où mes E_i sont des diviseurs exceptionnels, et $a_i \in \mathbb{Q}$ (les a_i peuvent ne pas être entiers car l'identité précédente est à comprendre comme $mK_{X'} = \pi^*(mK_X) + \sum_i ma_iE_i$ et a priori seulement les ma_i sont entiers).

Les notions suivantes, initiées par Miles Reid, ont vu leur importance s'accroître parallèlement aux avancées du programme du modèle minimal (cf [**KM98**]) :

Définition 1.4.1. — Soit X une variété normale $\mathbb{Q}$-Gorenstein. On dit que X est à singularités terminales (resp. canoniques) si pour toute résolution $\pi : X' \to X$, les discrépances a_i vérifient $a_i > 0$ (resp. $a_i \geqslant 0$).

Il est facile de voir qu'il suffit de vérifier ces conditions sur *une* résolution. Par ailleurs, la terminologie provient du fait que si X est de type général (ie K_X big) et si l'algèbre $R(X, K_X) := \bigoplus_{m \geqslant 0} H^0(X, mK_X)$ est de type fini, alors $\text{Proj}(R(X, K_X))$ est à singularités canoniques. Notons que cette condition ($R(X, K_X)$ de type fini) est réalisée si K_X est semi-ample, ou encore si X est lisse d'après [**BCHM10**], comme nous allons l'expliquer dans le paragraphe suivant.

1.4.2. Finitude de l'anneau canonique. — Avant de mentionner les résultats récents du MMP, il nous faut parler du cadre naturel que constituent les paires. Une paire (X, D) est constituée d'une variété normale (projective) X et d'un $\mathbb{Q}$-diviseur de Weil appelé bord. Beaucoup de raisons, dont la majorité tournent autour de la formule d'adjonction font qu'il est pratique en géométrie birationnelle de considérer des paires, notamment afin de pouvoir faire des raisonnements par récurrence en utilisant le (ou un) bord. Par ailleurs, étant donné que le théorème d'Hironaka nous permet aussi de construire des log résolutions de paires, les notions "absolues" peuvent en général s'étendre sans problème au cas des paires. Rappelons qu'une log résolution d'une paire (X, D) est un morphisme birationnel $\pi : X' \to X$ tel que X' est lisse, et tel que D', transformé strict de D, est une diviseur dont le support est à croisements normaux simples. De plus, on peut toujours s'arranger pour que π soit un isomorphisme au-dessus du lieu où la paire (X, D) est log lisse (ie X lisse et D à support snc).

On suppose maintenant que $K_X + D$ est $\mathbb{Q}$-Cartier, et on le tire en arrière par π, ce qui donne :

$$K_{X'} = \pi^*(K_X + D) + \sum_i a_i E_i$$

où cette fois, E_i est soit un diviseur exceptionnel, soit une composante de D'. On donne alors la définition suivante :

Définition 1.4.2. — Soit (X, D) une paire telle que $K_X + D$ est $\mathbb{Q}$-Cartier. On dit que (X, D) est une paire Kawamata log terminale (resp. log canonique) si pour toute résolution $\pi : X' \to X$, les discrépances a_i vérifient $a_i > -1$ (resp. $a_i \geqslant -1$).

A nouveau, il suffit de vérifier la condition sur une résolution. Notons qu'on abrège en général les notations en écrivant klt ou lc. Par ailleurs, on voit que les coefficients d'un bord D d'une paire klt (resp. lc) a nécessairement des coefficients < 1 (resp. $\leqslant 1$). Il est intéressant de garder à l'esprit que la condition "klt" est une condition de volume fini pour une certaine forme volume (méromorphe) naturellement attachée à la paire (X, D), cf le paragraphe suivant par exemple.

Bien que ce ne soit pas évident au premier abord, il y a une véritable différence entre les paires klt et les paires lc. Par exemple, une variété à singularité log terminales (ie $(X, 0)$ est klt) a des singularités rationnelles alors que ce n'est pas le cas en général pour les variétés lc. Ensuite, le théorème du base-point-free de Shokurov-Kawamata (K_X nef et big implique K_X semi-ample) est vrai dans le cas klt mais faux en général dans le cas lc. La classe log canonique est la plus grande classe où les discrépances ont un vraiment du sens. Et surtout, on dispose du théorème fondamental, qui reste une des conjectures majeures en géométrie birationnelle dans le cas log canonique :

Théorème 1.4.3 (**[BCHM10]**). — *Soit (X, D) une paire klt telle que $K_X + D$ est $\mathbb{Q}$-Cartier. Alors l'anneau*

$$R(X, K_X + D) = \bigoplus_{m \in \mathbb{N}} H^0(X, \lfloor m(K_X + D) \rfloor)$$

est de type fini. Si de plus $K_X + D$ est big, alors $K_X + D$ a un modèle log canonique.

Ce que veut dire la dernière phrase est la chose suivante : $X_{\text{can}} := \text{Proj}(R(X, K_X + D))$ est bien une variété projective normale. D'autre part, l'application de Kodaira $f : X \dashrightarrow \mathbb{P}^N$ induite par (une grande puissance de) $K_X + D$ induit un morphisme birationnel sur son image qui est naturellement isomorphe à X_{can}. Enfin, en notant $D_{\text{can}} := f_* D$, alors la paire $(X_{\text{can}}, D_{\text{can}})$ est klt et vérifie que $K_{X_{\text{can}}} + D_{\text{can}}$ est *ample*. Enfin, si $\mu : Y \to X$, $\nu : Y \to X_{\text{can}}$ est une résolution du graphe de f, alors

$$\mu^*(K_X + D) = \nu^*(K_{X_{\text{can}}} + D_{\text{can}}) + E$$

pour un diviseur E effectif et ν-exceptionnel. Le fibré $K_X + D$ admet donc une décomposition de Zariski au sens fort.

1.4.3. Pourquoi construire des métriques KE sur des variétés singulières ?— Bonne question ! Mais avant de répondre à cette question, il faut savoir au moins un minimum ce qu'il faut attendre d'une telle métrique : quelle sorte d'objet est-ce censé être ? Tout d'abord, la moindre des choses à demander, c'est d'avoir sur la partie régulière $X_{\rm reg}$ de la variété une métrique de Kähler-Einstein $\omega_{\rm KE}$ au sens habituel. Bien sûr, une telle condition sur une variété ouverte n'est pas assez restrictive et il faut d'une manière ou d'une autre imposer une condition au bord (c'est-à-dire près des singularités, ou près du diviseur exceptionnel si l'on travaille sur une log résolution) s'il on veut obtenir un objet raisonnable.

Sur une variété normale, $\omega_{\rm KE}$ va s'étendre automatiquement sur X tout entier en un courant positif fermé dès lors que sa classe de cohomologie provient d'une classe α sur X. Remarquons que cette condition est automatique dans le cas de courbure égale à 1, avec $\alpha = -K_X$ (resp. égale à -1, avec $\alpha = K_X$).

Une condition de bord assez naturelle est alors de demander que $\omega_{\rm KE}$ ne porte pas de masse sur $X_{\rm sing}$, ou autrement dit que $\int_{X_{\rm reg}} \omega_{\rm KE}^n = \alpha^n$; ainsi les potentiels locaux vont vivre dans des espaces d'énergie finie pour lesquels la théorie pluripotentialiste est très efficace. Notons que cette condition de masse (ainsi que la condition KE) est intrinsèque sur $X_{\rm reg}$, et donne ainsi une définition de métrique de Kähler-Einstein assez naturelle à manipuler. Enfin, mentionnons que si X est à singularités log terminales, alors la condition de masse impose que les potentiels locaux soient continus sur X tout entier (cf [**EGZ09, BBE+11**]), la réciproque étant par ailleurs une conséquence directe de la théorie de Bedford et Taylor.

Observons pour finir qu'on ne peut pas raisonnablement exiger que le courant soit lisse sur toute la variété, autrement dit qu'il soit localement (dans des plongement locaux de X dans $\mathbb{C}^N$) la restriction de formes lisses. En effet, pensons déjà au cas orbifold, où l'on s'attend à ce que la métrique soit lisse au sens orbifold, c'est-à-dire une fois lue dans les cartes locales uniformisantes, ce qui ne coïncide pas avec la lissité évoquée plus haut.

On peut aussi se demander à quoi ressemble un tel objet une fois remonté sur une log résolution. On définit D' par $K_{X'} + D' = \pi^*(K_X + D)$ (donc D' n'est pas le transformé strict de D). Alors, sur la log résolution (X', D'), on voudrait donc avoir une métrique kählerienne à courbure de Ricci constante hors du support de D', mais dont on ne sache pas forcément ce qu'il se passe près de D'. Mais dans les parties précédentes, on a (presque) expliqué ce que devait être une métrique de Kähler-Einstein pour une paire log lisse : c'est un courant positif fermé ω' dont la forme de Ricci est bien définie comme courant, et vérifiant $\mathrm{Ric}\,\omega' = \lambda\omega' + [D']$ pour un certain $\lambda \in \mathbb{R}$. Voilà donc un candidat, disons au moins s'il avait le bon goût de provenir d'un courant sur la variété singulière de départ X'.

Il se trouve qu'une telle définition est bien intrinsèque, et équivalente à la définition (chronologiquement postérieure) expliquée plus haut en terme de masse, étant donnés les travaux fondateurs de [**EGZ09**] (courbure négative ou nulle) puis [**BBE+11**] (courbure positive). Partons d'une paire (X, D) telle que $K_X + D$ est $\mathbb{Q}$-Cartier et ample (resp. trivial, anti-ample). On se place en un point $x \in X_{\rm sing}$, on considère une trivialisation locale (dans un voisinage de x dans $X_{\rm reg}$) σ de $m(K_X + D)$, puis une métrique hermitienne h sur $\pm(K_X + D)$ (il n'y a rien à faire dans le cas plat) dont la courbure est une forme kählerienne ω_0. On introduit alors la mesure $\mu := \frac{(\sigma\wedge\bar\sigma)^{1/m}}{|\sigma|_h^{2/m}}$ qui se trouve être indépendante de la trivialisation et est bien définie sur $X_{\rm reg}$, puis sur X en la prolongeant par 0 sur $X_{\rm sing}$. On considère alors l'équation

$$(\omega_0 + dd^c\varphi)^n = e^{\lambda\varphi}\mu$$

pour un certain $\lambda \in \mathbb{R}$, le produit étant à considérer au sens non-pluripolaire. Une solution de cette équation sera par définition une métrique de Kähler-Einstein pour la paire (X, D).

Plusieurs observations s'imposent. Tout d'abord, sur tout ouvert de $X_{\rm reg}$ où $\omega = \omega_0 + dd^c\varphi$ sera lisse, elle induira une métrique de Kähler-Einstein au sens usuel. D'autre part, si on considère une log résolution (X', D') de la paire (X, D), alors la métrique $\omega' = \pi^*\omega$ sera naturellement une métrique de Kähler-Einstein pour (X', D'). Ainsi, si (X, D) est d'ores et déjà log lisse avec D à coefficients dans $[0, 1]$, alors on retrouve bien la notion de métrique Kähler-Einstein manipulée dans les parties précédentes, et les singularités de cette métrique sont alors coniques ou Poincaré (au moins conjecturalement).

On dispose maintenant d'une définition en bonne et due forme (on verra par la suite qu'on peut la remplacer assez avantageusement par une définition n'impliquant que la donnée de la métrique sur $X_{\rm reg}$, mais oublions cela pour le moment). Passons également outre la construction de telles métriques, qui sera évoquée dans le paragraphe suivant, et penchons nous sur la question de l'intérêt de tels objets.

Pour commencer, il y a un intérêt théorique assez manifeste : celui de construire des objets issus de la géométrie différentielle dans le cadre de la géométrie algébrique. Dans cette même logique, les conséquences des théorèmes d'Aubin et Yau en géométrie (complexe et algébrique) ont été extrêmement nombreuses ; pour n'en citer que quelques unes, on peut mentionner la classification des surfaces homotopes à $\mathbb{P}^2$, le théorème de Beauville-Bogomolov ou plus généralement la théorie des variétés Calabi-Yau, la polystabilité du fibré tangent des variétés canoniquement polarisées etc. On peut alors se douter (ou tout du moins espérer) que la construction de métriques de Kähler-Einstein singulières aura également des répercussions intéressantes en géométrie algébrique. D'ailleurs, nous expliquerons plus loin une application assez explicite en termes de semi-stabilité du faisceau tangent des variétés log canoniques.

Enfin, les récentes avancées concernant la construction de métriques KE sur les variétés de Fano ont montré l'importance de considérer des limites (au sens de Gromov-Hausdorff) de variétés KE, et que ces limites étaient naturellement équipées d'une structure de variété projective normale à singularités log terminales ainsi que d'une métrique KE au sens expliqué plus haut, cf. [**DS12, CDS12b**].

1.4.4. Construction des métriques singulières. — Dans cette partie, nous allons expliquer sous quelles conditions sur les singularités de la variété on sait construire des métriques KE, en détaillant un peu les difficultés qui apparaissent dans ce nouveau cadre. Une observation de [**BBE$^+$11**, Proposition 3.8] montre qu'une variété normale admettant sur un ouvert de Zariski $\Omega \subset X_{\rm reg}$ une métrique kählerienne à courbure de Ricci constante positive ou nulle a nécessairement des singularités au pire log terminales. En courbure négative, la situation est un peu différente au sens où les plus mauvaises singularités possibles sont les singularités log canoniques, comme il a été montré dans [**BG13**]. Ainsi, restreignons-nous dans un premier temps à des variétés klt, ou disons plutôt des paires klt (la situation lc est différente sous beaucoup d'aspects).

Supposons par exemple que la courbure soit négative, ie $K_X + D$ est ample ; on fixe ω_0 une métrique kählerienne dans cette classe. On considère une log résolution (X', D'), ie $K_{X'} + D' = \pi^*(K_X + D)$. Alors l'équation de Monge-Ampère va prendre la forme

$$(\mathbf{MA}') \qquad (\pi^*\omega_0 + dd^c\varphi)^n = \frac{e^{\varphi} dV}{\prod_i |s_i|^{2a_i}}$$

où dV est une forme volume sur X', et $D' = \sum a_i D_i$ est un diviseur à support snc dont les coefficients vérifient $a_i < 1$ par la condition que (X, D) est klt.
La nouvelle difficulté pour résoudre cette équation est le fait que la solution vit dans une classe qui n'est plus kählerienne mais seulement semi-ample et big. Pour surmonter cette difficulté, il a fallu comprendre que l'estimée de Kołodziej était préservée lorsque faisait tendre une classe kählerienne vers une classe semi-ample et big, ce qui a été fait par [**EGZ09**] et [**Zha06, TZ06**]. Une fois l'équation résolue (au moins au sens faible) avec une estimée $\mathscr{C}^0$, la lissité de la solution hors du support de D' peut s'obtenir en utilisant l'astuce de Tsuji [**Tsu88b**].

A ce stade, on sait donc que les paires klt (X, D) telles que $K_X + D$ est ample admettent une unique métrique de Kähler-Einstein, et que cette métrique est lisse sur $X_{\text{reg}} \setminus \text{Supp}(D)$ et que son potentiel est globalement borné. En dehors de ces informations là, on ne sait à l'heure actuelle pas grand chose de plus. Un des grands défis est de comprendre le comportement asymptotique de cette métrique près des singularités de X, mais cela semble encore complètement hors de portée. Dans la section suivante, nous verrons qu'on peut dire quelque chose au voisinage de certains points de D.

Pour finir cette section, mentionnons un problème très intéressant, à savoir celui de construire des métriques de Kähler-Einstein dans des classes big. Par exemple, prenons une variété lisse de type général, ie telle que K_X est big. Peut-on construire une métrique KE sur X qui soit au moins lisse sur le lieu ample de K_X ? Cette question a été beaucoup étudiée, par exemple par [**Tsu88b**] ou [**Sug90**] qui ont apporté une réponse affirmative dans le cas où la variété est minimale, c'est-à-dire si K_X est nef. Dans ce cas, on peut ramener la question de la lissité à un problème d'estimées dans le cas kählerien.

Du point de vue des équations de Monge-Ampère (dans des classes big donc), le formalisme pour de telles questions a été élaboré dans [**BEGZ10**] dans un grand degré de généralité, et où est traitée la régularité dans le cas nef et big. A ce jour, la question de la régularité dans le cas big pour des équations de Monge-Ampère complexes est encore ouverte.

Mais dans le cas particulier où l'on s'intéresse à des métriques KE à courbure négative sur des variétés de type général, ou plus généralement sur des paires klt (X, D) de log type général (ie telles que $K_X + D$ est big), alors les résultats profonds de [**BCHM10**] rappelés plus haut montre que la métrique KE de (X, D) peut s'obtenir à partir d'un modèle log canonique $(X_{\text{can}}, D_{\text{can}})$ où $K_{X_{\text{can}}} + D_{\text{can}}$ est ample. Il n'est alors pas difficile d'en déduire que la métrique KE sur (X, D) est lisse sur $X_{\text{reg}} \setminus \text{Supp}(D)$ intersecté avec le lieu ample de $K_X + D$.

1.4.5. Paires klt et singularités coniques. — Dans ce paragraphe, on va expliquer brièvement les résultats de [**Gue12a**], qui seront détaillés ensuite au chapitre 3. La problématique est la suivante : étant donnée (X, D) une paire klt (D effectif) dont on suppose qu'elle admet une métrique de Kähler-Einstein ω_{KE} (typiquement $K_X + D$ est ample ou même seulement big, mais la discussion s'appliquera aussi aux variétés log Fano qui sont Kähler-Einstein), on se demande à quoi ressemble la métrique en question. Comme on l'a rappelé plus haut, on sait que ω_{KE} est lisse sur $X_{\mathrm{reg}} \setminus \mathrm{Supp}(D)$, et qu'elle vérifie sur X_{reg} :

$$\mathrm{Ric}\,\omega_{\mathrm{KE}} = \lambda\omega_{\mathrm{KE}} + [D]$$

Par ailleurs, on sait également grâce à [**EGZ09, Zha06**] que le potentiel de ω_{KE} est globalement borné sur X. Que peut-on dire de plus ? Près de X_{sing}, la situation est quasiment complètement incomprise même dans le cas où $D = 0$, et il semble difficile d'obtenir des informations précises à cet endroit. De même, là où le diviseur est singulier (ou disons là où il n'est pas snc) on ne connaît pas de modèle local pour ω_{KE}, donc on va se heurter aux mêmes difficultés. En revanche, là où la paire (X, D) est log lisse, c'est-à-dire là sur l'ouvert de Zariski où X est lisse et D snc, il existe un modèle pour ω_{KE}, à savoir le modèle conique. De plus, sur ce lieu, ω_{KE} vérifie la même équation de Kähler-Einstein qu'une métrique KE pour une paire log lisse klt.

Ainsi, la question est de savoir si la régularité conique est "locale", comme l'est la régularité usuelle. Autrement dit, partant d'une équation de Monge-Ampère (dégénérée) globale, est-ce que la solution se comporte de la même manière sur le lieu log lisse qu'une solution de l'équation globale associée au cas log lisse ? La question n'est pas gratuite, car on sait qu'il n'y a pas de régularité pour l'équation de Monge-Ampère locale (cf [**Pog71**] ou [**Bło99**]). L'argument doit donc nécessairement être global. Donnons maintenant l'énoncé :

Théorème 1.4.4 **(**[**Gue12a**]**).** — *Soit (X, D) une paire klt, et soit $\mathrm{LS}(X, D) := \{x \in X; (X, D)$ est log lisse en $x\}$. On suppose que les coefficients de D sont dans $[1/2, 1)$.*

(*i*) *Si $K_X + D$ est big, alors la métrique de Kähler-Einstein de (X, D) est à singularités coniques le long de D sur $\mathrm{LS}(X, D) \cap \mathrm{Amp}(K_X + D)$.*
(*ii*) *Si $-(K_X + D)$ est ample, alors toute métrique de Kähler-Einstein pour (X, D) a des singularités coniques le long de D sur $\mathrm{LS}(X, D)$.*

On renvoie donc à l'article original ou au chapitre 3 pour la preuve de ce théorème, mais on peut quand même essayer de souligner les difficultés et mentionner les techniques pour surmonter ces dernières.

Pour commencer, expliquons comment ramener le cas big au cas ample en supposant qu'on sait démontrer le théorème dans ce cas. Grâce aux résultats de [**BCHM10**], on dispose d'une application birationnelle $f : X \dashrightarrow X_{\mathrm{can}}$ et d'un diviseur non trivial $D_{\mathrm{can}} := f_* D$ telle que la paire $(X_{\mathrm{can}}, D_{\mathrm{can}})$ est klt et telle que $K_{X_{\mathrm{can}}} + D_{\mathrm{can}}$ est ample. Ainsi, on peut construire sur X_{can} une métrique KE ω_{can} à courbure négative, qui aura des singularités coniques le long de D_{can} sur le lieu log lisse de la paire. Par ailleurs, l'application f n'est pas quelconque, mais c'est l'application de Kodaira associée à un grand multiple (fixe) de $K_X + D$, et il est facile de voir qu'une telle application induit un isomorphisme quand on la restreint au lieu ample de $K_X + D$. Ainsi, sur ce lieu, $\omega_{\mathrm{KE}} = f^*\omega_{\mathrm{can}}$ a le même comportement que ω_{can} et donc en se retreignant ensuite au lieu log lisse, on obtient bien le résultat souhaité.

On peut donc maintenant supposer que $K_X + D$ est ample (on laisse le cas log Fano de coté). La première chose à faire est de choisir une log résolution $\pi : (X', D') \to (X, D)$ qui ne touche pas au lieu log lisse (on sait qu'une telle log résolution existe bien). Du coté Monge-Ampérien, on se retrouve alors avec une équation de la forme

$$(\pi^*\omega_0 + dd^c\varphi)^n = \frac{e^{\varphi}dV}{\prod_i |s_i|^{2a_i}}$$

où ω_0 est une métrique kählerienne dans $c_1(K_X + D)$, dV est une forme volume sur X', et $D' = \sum a_i D_i$ est un diviseur à support snc dont les coefficients vérifient bien sûr $a_i < 1$.

Bien sûr, il faut isoler la partie qui vient de D de la partie exceptionnelle. On va donc noter t_i les sections des diviseurs exceptionnels, et laisser s_i désigner les sections des composantes du transformé strict de D. On écrit alors

$$(\mathbf{MA}') \qquad (\pi^*\omega_0 + dd^c\varphi)^n = \prod_j |t_j|^{2b_j} \frac{e^{\varphi}dV}{\prod_i |s_i|^{2a_i}}$$

où les b_j satisfont cette fois $b_j > -1$ (il peut y avoir des pôles malgré la notation). Comme vous aurez deviné, on va régulariser la partie conique :

$$(\mathbf{MA}'_\varepsilon) \qquad (\pi^*\omega_0 + dd^c\varphi_\varepsilon)^n = \prod_j |t_j|^{2b_j} \frac{e^{\varphi_\varepsilon}dV}{\prod_i (|s_i|^2 + \varepsilon^2)^{a_i}}$$

et il nous faut ensuite trouver une métrique de référence ω_ε avec des singularités coniques (approchées) le long du transformé strict de D. C'est là où les choses se compliquent un peu, car on veut rester dans la classe de $\pi^*\omega_0$ qui n'est pas kählerienne. On peut toujours ajouter le potentiel conique habituel, disons ψ_ε à $\pi^*\omega_0$, qui aura les bonnes singularités hors du lieu exceptionnel, mais alors ce n'est plus clair du tout que la courbure bisectionnelle holomorphe de cette métrique va rester borner inférieurement désormais. Or cette propriété est essentielle pour nous afin d'utiliser une inégalité de laplacien.

La solution est alors d'éclater à nouveau X' pour rendre le lieu exceptionnel E divisoriel afin de disposer d'une fonction de la forme $\chi = \delta \log |s_E|^2$ telle que $\pi^*\omega_0 - dd^c\chi$ soit hors du lieu exceptionnel la restriction d'une forme kählerienne globale. Alors, en utilisant cette métrique comme référence, on constate que la métrique conique régularisée a alors bien une courbure bisectionnelle minorée, puis en adaptant l'estimée de [**BBE$^+$11**, Theorem 10.1] à notre contexte, on peut alors réutiliser les arguments de [**CGP11**] pour arriver à nos fins. L'idée est donc assez simple, mais les détails sont plutôt techniques comme on peut s'en rendre compte en lisant attentivement [**Gue12a**].

1.4.6. Variétés slc et métriques KE. — Nous expliquons maintenant les résultats de notre article [**BG13**] en collaboration avec Robert Berman.

La question de départ est la suivante : reste-t-il des variétés singulières sur lesquelles on ne sache pas encore construire de métriques de Kähler-Einstein ? D'après les travaux de [**EGZ09**] et [**BBE$^+$11**], on connait la réponse pour les variétés log terminales, ou plus généralement pour les paires klt. Peut-on aller au delà de cette classe de singularités ? D'après [**BBE$^+$11**, Proposition 3.8], une log paire (X, D) admettant sur X_{reg} une métrique KE à potentiels continus et de courbure positive ou nulle est nécessairement klt. Ainsi, l'histoire s'arrête là en courbure positive ou nulle. Reste la courbure négative. On sait déjà qu'une paire klt (X, D) telle que $K_X + D$ est ample admet une métrique KE. Cependant, une paire

(X, D) peut admettre une métrique KE sans être klt, comme on l'a vu dans le cas des paires log lisses avec des métriques KE à singularités Poincaré.

Ainsi, on est naturellement amené à se demander si une paire log canonique (X, D) telle que $K_X + D$ est ample peut admettre une métrique KE. Et inversement si une telle paire admet une métrique KE, est-elle nécessairement log canonique ? En fait, on peut élargir un peu la question pour la rendre encore plus pertinente : peut-on munir les variétés canoniquement polarisées à singularités semi-log canoniques (slc) d'une métrique KE ?

1.4.6.1. Variétés semi-log canoniques. — La dernière phrase mérite des explications ; elles arrivent. Mais avant cela, parlons déjà un peu de singularités log canoniques. En dimension deux par exemple, on dispose d'une classification complète en termes de la résolution minimale : la singularité doit être elliptique simple (ie le diviseur exceptionnel est une courbe elliptique lisse), un cusp (le diviseur exceptionnel est un cycle de courbes rationnelles), ou bien un quotient par un groupe fini de telles singularités. Par exemple, un cône sur une courbe elliptique est à singularités log canoniques mais pas log terminales.

Une autre large classe d'exemples est donnée par les variétés arithmétiques. En effet, Alexeev a observé [**Ale96**] qu'étant donnée une variété torique normale X et le diviseur B correspondant aux faces de codimension 1 de son éventail, alors la paire (X, B) est log canonique. Soit maintenant Γ un groupe arithmétique net (ie les valeurs propres de chaque élément de Γ engendre un sous-groupe sans torsion de $\mathbb{C}^*$) agissant sur un domaine symétrique borné D ; alors D/Γ est lisse. On considère maintenant $(D/\Gamma)^*$ la compactification de Baily-Borel, et $\overline{D/\Gamma}$ une compactification toroïdale quelconque ; on note Δ^* et $\overline{\Delta}$ les diviseurs de bords respectifs. Alors Alexeev déduit de l'observation précédente et des résultats de Mumford [**Mum77**, 3.4,4.2] que $((D/\Gamma)^*, \Delta^*)$ est le modèle log canonique de $(\overline{D/\Gamma}, \overline{\Delta})$ et que ces deux paires ont des singularités log canoniques. Quant au cas d'un groupe arithmétique quelconque, il se ramène au cas précédent en rajoutant un bord orbifold, cf [**Ale96**, Theorem 3.4].

Expliquons maintenant pourquoi il est naturel de sortir (légèrement) du cadre des variétés normales. La raison principale apparaît quand on regarde des familles de variétés canoniquement polarisées et leur dégénérescences éventuelles. Viehweg [**Vie95**] a montré qu'il existait un espace de modules quasi-projectif des variétés lisses X telles que K_X est ample (et à polynôme de Hilbert fixé disons). La question suivante est de savoir si on peut rajouter certaines variétés (et le cas échéant lesquelles) afin de compactifier cet espace de modules. Ce problème a été intensivement étudié par de nombreuses personnes parmi lesquelles Alexeev, Kollár, Shepherd-Barron [**KSB88**], Karu [**Kar00**], Kovács, etc. mais encore aujourd'hui la situation n'est pas complètement claire. Ce qui émerge de ces travaux est la notion de variété stable, qui semble être la bonne classe de variétés à rajouter à notre espace de modules pour le compactifier. Mais comme on s'en aperçoit dès la dimension un, ces variétés ne sont pas normales en général, car elles ont des singularités en codimension un. Sans vouloir rentrer dans les détails techniques de la définition de variété stable, on peut les voir comme des variétés à canonique ample et dont les seules singularités non-normales sont de type $\{z_1 z_2 = 0\} \subset \mathbb{C}^{n+1}$, et telles que la paire obtenue en prenant la normalisée de X d'une part et le transformé strict via cette normalisation du diviseur singulier, est une paire à singularités log canoniques. Ces singularités s'appellent singularités *semi-log* *canoniques*. Notons que de telles variétés ne sont pas irréductibles en général. On renvoie à [**Kov12, Kol10, Kol**] pour un aperçu complet de la question, ou à [**BG13**, §2.3] pour une introduction un peu plus synthétique.

1.4.6.2. Métriques KE. — Revenons à nos moutons kähleriens. On voudrait désormais équiper ces variétés singulières canoniquement polarisées (log canoniques, ou semi-log canoniques plus généralement) d'une métrique de Kähler-Einstein. Mais on est maintenant dans un cadre non-normal où l'on n'a pas encore défini la notion de métrique KE. On pourrait se ramener au cadre usuel en définissant tout à partir de la normalisée, mais pour plusieurs raisons, on préfère donner la définition suivante, dont on montrera par la suite qu'elle est équivalente à toutes les autres définitions raisonnables qu'on peut imaginer (on renvoie aux références précédemment citées pour les termes techniques) :

Définition 1.4.5. — Soit X une variété projective satisfaisant les conditions G_1 et S_2, et telle que K_X est ample. Une métrique de Kähler-Einstein pour X est une métrique kählerienne ω sur X_{reg} telle que :

1. $\mathrm{Ric}\,\omega = -\omega$ sur X_{reg},
2. $\int_{X_{\mathrm{reg}}} \omega^n = c_1(K_X)^n$.

Commençons par quelques remarques dans le désordre. La première condition est très naturelle, mais la seconde est moins évidente à justifier. Il faut clairement une condition au bord, ne serait-ce que pour l'unicité ; en fait cette condition de masse est là essentiellement pour garantir que les potentiels avec lesquels on va travailler sont dans les bonnes classes d'énergie dans lesquelles on dispose d'outils efficaces pour analyser les équations de Monge-Ampère (au sens non pluripolaire). Ce qui est agréable avec cette définition, c'est qu'elle ne fait intervenir que le donnée du courant sur la partie lisse de X, et que la donnée globale de X n'apparaît qu'à travers l'information numérique de l'intersection maximale de K_X. A cause des singularités éventuelles en codimension un, il y aura quelque chose de non trivial à vérifier pour voir qu'un tel courant s'étend automatiquement à travers les singularités de X.

Théorème 1.4.6. — *Soit X une variété projective satisfaisant les conditions G_1 et S_2, et telle que K_X est ample. Alors X admet une métrique de Kähler-Einstein si et seulement si X est à singularités semi-log canoniques.*

Par ailleurs, une telle métrique est toujours unique. Ce résultat répond donc complètement à la question posée de savoir quelle est la classe maximale de variétés qu'on peut munir de métriques de Kähler-Einstein. Avant d'expliquer rapidement la preuve, donnons quelques applications intéressantes. La première est la caractérisation des variétés KE en termes de K-stabilité, répondant ainsi positivement à la conjecture de Yau-Tian-Donaldson dans le cas singulier, canoniquement polarisé. C'est une conséquence immédiate des travaux d'Odaka [**Oda08, Oda11**] et du théorème ci-dessus :

Théorème 1.4.7. — *Soit X une variété projective satisfaisant les conditions G_1 et S_2, et telle que K_X est ample. Alors X admet une métrique de Kähler-Einstein si et seulement si (X, K_X) est K-stable.*

La condition de K-stabilité signifie que pour toute configuration test $\mathfrak{X} \to \mathbb{C}$, l'invariant de Donaldson-Futaki est toujours positif, et s'annule si et seulement si la configuration est la configuration triviale $X \times \mathbb{C}$. On renvoie aux articles d'Odaka pour plus de détails là-dessus.

Une autre application concerne la finitude du groupe d'automorphismes de variétés log canoniques :

Théorème 1.4.8. — *Soit X une variété normale à singularités log canoniques. Si K_X est ample, alors* $\mathrm{Aut}(X)$ *est fini.*

En effet, comme $\mathrm{Aut}(X)$ préserve la polarisation K_X, ce dernier est un groupe algébrique. Par ailleurs, son algèbre de Lie est donnée par les champs de vecteurs $H^0(X, \mathscr{T}_X)$, mais par normalité, ce dernier espace est naturellement isomorphe à $H^0(X_{\mathrm{reg}}, T_{X_{\mathrm{reg}}})$. Tout revient donc à montrer que X_{reg} n'admet pas de champ de vecteurs global.

Mais en utilisant l'unicité de la métrique KE, on voit que la dérivée de Lie de ω_{KE} suivant les parties réelles et imaginaires d'un tel champ de vecteurs est nulle. En manipulant convenablement les relations entre ces deux champs de vecteurs, on aboutit à leur nullité. Ainsi la preuve est une conséquence facile de l'existence et l'unicité de la métrique KE. Ce résultat a également été obtenu par [**BHPS12**] de manière purement algébrique en utilisant le théorème de Bogomolov-Sommese généralisé aux variétés log canoniques par [**GKKP11**].

Donnons maintenant une idée de la preuve du Théorème 1.4.6. Il y a deux parties indépendantes. La première est la construction de la métrique dans le cas semi-log canonique ; l'autre est de voir que l'existence d'une telle métrique impose les singularités attendues.

Commençons par cette dernière. On part donc d'une métrique KE $\omega = \omega_0 + dd^c\varphi$ pour une métrique kählerienne $\omega_0 \in c_1(K_X)$. Elle va automatiquement s'étendre à X (il y a un argument non-trivial dans le cas non normal, donné à la [**BG13**, Proposition 2.6]). La condition de masse garantit que le potentiel φ appartient à l'espace $\mathcal{E}(X, \omega_0)$. D'après les résultats de [**BFJ12**] utilisés dans le contexte pluripotentialiste [**BBE+11**, Appendix A], on en déduit que pour toute log résolution de la normalisation $\pi : X' \to X^\nu$, $\pi^*\varphi^\nu$ n'a aucun nombre de Lelong. Comme l'équation de Kähler-Einstein se traduit sur X' en une équation de Monge-Ampère du type

$$(\pi^*\omega_0 + dd^c\varphi)^n = \frac{e^\varphi dV}{\prod_i |s_i|^{2a_i}}$$

et que $\int_{X'} (\pi^*\omega_0 + dd^c\varphi)^n < +\infty$, il n'est pas trop difficile d'en déduire que tous les a_i vérifient $a_i \leqslant 1$, et donc que X est bien à singularités semi-log canoniques.

L'autre direction demande largement plus d'efforts. Tout d'abord, il faut produire une solution faible à l'équation de Monge-Ampère ci-dessus. Pour ce faire, on adapte la méthode variationnelle développée dans [**BBGZ09**] qui est assez souple pour travailler dans un contexte dégénéré. La nouvelle difficulté qui apparaît ici est que la forme volume μ qu'on va multiplier par e^φ n'est plus de masse finie, et il faut donc toujours travailler dans des espaces de potentiels φ tels que $\int e^\varphi d\mu < +\infty$. Mais construire la solution faible ne suffit pas, il faut aussi montrer sa lissité sur la partie régulière de X ; pour cela, on va établir des estimées a priori sur les équations de Monge-Ampère convenablement régularisées (ie on régularise la classe π^*K_X en une classe ample, et on régularise le membre de droite de l'équation de MA). En supposant ces estimées établies, on aura aussi besoin de savoir que les solutions de l'équation approchée convergent bien vers notre solution. Ces deux choses représentent beaucoup de travail à vrai dire. Il est difficile d'expliquer vraiment ce qu'il se passe en quelques lignes, mais on peut déjà souligner que ne serait-ce que pour obtenir une estimée d'ordre zéro sur le potentiel, on a besoin d'introduire des fonctions barrières de divers types (essentiellement log et log log) puis d'utiliser le contexte introduit par Kobayashi [**Kob84**] et Tian-Yau [**TY87**] pour traiter le cas purement log lisse.

Pour conclure, mentionnons que le même théorème d'existence de métrique de Kähler-Einstein reste vrai en supposant seulement K_X nef et big, et que l'on arrive également à obtenir l'existence de la métrique KE (au sens faible) lorsque X est de type général (K_X big).

1.5. Polystabilité du faisceau tangent

1.5.1. Notions de stabilité. — La notion de stabilité est une notion très vaste et riche, et le mot stabilité peut être employé dans énormément de contextes différents (systèmes dynamiques, algèbre linéaire, géométrie algébrique, théorie des nombres, etc.). Même à l'intérieur d'un domaine précis (la géométrie algébrique dans notre cas), le même mot peut receler différents sens. En ce qui nous concerne, la notion trouve ses sources dans la théorie géométrique des invariants (GIT en anglais) où l'on cherche à comprendre le comportement de certaines familles de variétés, ou plus généralement d'espaces de modules d'objets comme des faisceaux par exemple. Dans le premier cas, on fait opérer sur une variété projective un sous groupe du groupe projectif linéaire, typiquement un de ses sous-groupes à paramètres $\mathbb{C}^*$, et on regarde ce qu'il se passe lorsque le paramètre tend vers 0. Cela donne alors lieu à différentes notions de stabilité (qui ne sont pas sans rappeler la notion de K-stabilité liée aux configurations tests de variétés polarisées dont on a brièvement parlé à la section 1.4.6.2).

Inspirés par ces idées, et afin de comprendre quels fibrés vectoriels de rang et classes de Chern fixés formaient une famille bornée, Mumford [**Mum63**] puis Takemoto [**Tak72**] ont introduit une notion appelée stabilité de pente ("slope stability") associée à tout fibré vectoriel (ou même faisceau cohérent sans torsion) sur une variété (normale) polarisée.

Pour un fibré vectoriel E, la définition de la pente (par rapport à une polarisation H fixée) est très simple : c'est juste l'intersection de la première classe de Chern de E avec H^{n-1}, le tout divisé par le rang de E. Dans le cas d'un faisceau, c'est un peu plus subtil. On va y venir, mais pour cela il faut rappeler quelques définitions.

Dans la suite, X est une variété complexe projective normale de dimension n, et $\mathscr{F}$ est un faisceau cohérent (ou encore un $\mathcal{O}_X$-module cohérent). On note $\mathscr{F}^* = \mathscr{H}om(\mathscr{F}, \mathcal{O}_X)$ le dual $\mathscr{F}$ et on dit que $\mathscr{F}$ est *réflexif* si le morphisme naturel

$$j : \mathscr{F} \to \mathscr{F}^{**}$$

est un isomorphisme. Ainsi, le dual d'un faisceau cohérent est toujours réflexif. De plus, étant donné que le noyau j est constitué exactement de la torsion de $\mathscr{F}$, un faisceau réflexif est

donc nécessairement sans torsion. Passons maintenant au rang de $\mathscr{F}$: c'est son rang au point générique (ou de manière équivalente son rang sur l'ouvert de Zariski où il est localement libre), on le note $\mathrm{rg}\,\mathscr{F}$.
La notion suivante est celle de déterminant d'un faisceau cohérent $\mathscr{F}$. On pose $r = \mathrm{rg}\,\mathscr{F}$, et on définit alors

$$\det \mathscr{F} := (\Lambda^r \mathscr{F})^{**}$$

comme étant le determinant de $\mathscr{F}$. C'est un faisceau réflexif de rang 1 sur X, mais à moins que X soit lisse (ou disons localement factoriel), il n'est pas localement libre, et ne provient donc pas d'un fibré en droites.
Comme X est normale, il y a une bijection entre les faisceaux réflexifs de rang un (à isomorphisme près) et les diviseurs de Weil (à équivalence rationnelle près).. Plus précisément , la correspondance marche ainsi : à un diviseur de Weil D, on associe $\mathcal{O}_X(D) = \{f, \mathrm{div}(f) \geqslant -D\}$ et dans l'autre sens, étant donné $\mathscr{F}$ un faisceau réflexif de rang 1, on choisit sur X_{reg} un diviseur (de Weil) représentant le fibré en droites $\mathscr{F}_{|X_{\mathrm{reg}}}$ et on prend son adhérence.
On notera alors $c_1(\mathscr{F})$ la classe d'équivalence de n'importe quel diviseur de Weil représentant $\det \mathscr{F}$. On peut enfin définir la pente de $\mathscr{F}$:

Définition 1.5.1. — Soit H un fibré en droites ample sur X. La pente de $\mathscr{F}$ par rapport à H est le nombre rationnel

$$\mu_H(\mathscr{F}) := \frac{c_1(\mathscr{F})\cdot H^{n-1}}{\mathrm{rg}\,\mathscr{F}}$$

Comme pour tout fibré vectoriel E, on a $c_1(E) = c_1(\det E)$, il est clair que cette définition généralise bien la définition évoquée plus haut pour les fibrés vectoriels. Alors, la notion de stabilité se définit comme suit :

Définition 1.5.2. — Soit $\mathscr{E}$ un faisceau cohérent sans torsion sur X, et H un fibré en droites ample.
On dit que $\mathscr{E}$ est *semi-stable* (resp. *stable*) par rapport à H si pour tout sous-faisceau cohérent non nul $\mathscr{F} \subset \mathscr{E}$ (resp. et propre), on a

$$\mu_H(\mathscr{F}) \leqslant \mu_H(\mathscr{E}) \quad (\text{resp.}\, \mu_H(\mathscr{F}) < \mu_H(\mathscr{E}))$$

On dit que $\mathscr{E}$ est *polystable* (par rapport à H) si $\mathscr{E}$ est somme directe de faisceaux stables de même pente.

Il est facile de voir qu'un faisceau polystable est bien semi-stable (on voit ainsi que l'hypothèse sur les pentes est importante).

1.5.2. Correspondance de Kobayashi-Hitchin. — La correspondance de Kobayashi-Hitchin est un théorème à la fois très joli et très profond au sens où il relie deux notions d'origine très différente. Il a également servi de modèle pour énoncer la conjecture de Yau-Tian-Donaldson dans le cadre des variétés de Fano admettant une métrique de Kähler-Einstein. L'énoncé est très simple, et affirme qu'un fibré vectoriel holomorphe E au dessus d'une variété kählerienne compacte (X, ω) admet une métrique d'Hermite-Einstein (par rapport à ω) si et seulement si E est polystable (par rapport à ω).

On peut rappeler qu'une métrique hermitienne h sur E est dite d'Hermite-Einstein par rapport à ω s'il existe une constante $\mu \in \mathbb{R}$ telle que

$$\textbf{(HE)} \qquad \Theta_h(E) \wedge \omega^{n-1} = \mu \mathrm{Id}_E \, \omega^n$$

ie que le tenseur de courbure de h contracté avec ω est proportionnel à l'identité.

Cette définition a été introduite par Kobayashi [**Kob80**] et généralise la notion de métrique de Kähler-Einstein au sens où si X admet une métrique de Kähler-Einstein ω, alors (T_X, ω) est Hermite-Einstein. Dans la correpondance de Kobayashi-Hitchin, le sens "HE implique polystable" est largement moins difficile, et a été prouvé par Kobayashi [**Kob82**] et Lübke [**Lüb83**] indépendamment. La preuve de l'autre direction a été initiée par Donaldson [**Don83**] (c'est d'ailleurs lui qui a attribué l'origine du problème à Kobayashi et Hitchin) lorsqu'il a prouvé le cas des surfaces de Riemann en redémontrant le théorème de Narasimhan-Seshadri [**NS65**] sur la correspondance entre les fibrés polystables de degré nul et les représentations unitaires de dimension finie du groupe fondamental. Il donna plus tard la preuve du cas des surfaces algébriques [**Don85**] puis du cas projectif en toute dimension [**Don87**]. Le cas kählerien général est lui dû à Uhlenbeck et Yau [**UY86, UY89**].

1.5.3. Semi-stabilité du faisceau tangent de variétés singulières. — Dans le cas du fibré tangent d'une variété lisse, la polystabilité est donc impliquée par l'existence d'une métrique d'une métrique de Kähler-Einstein, comme on l'a vu plus haut.

Pour les variétés canoniquement polarisées (à singularités log canoniques) on connaît déjà l'existence de la métrique de Kähler-Einstein (d'après Aubin et Yau [**Aub78, Yau78b**] dans le cas lisse, et [**BG13**] dans le cas de singularités log canoniques). Donc au moins dans le cas lisse, la polystabilité du tangent par rapport à la classe canonique est acquise d'après [**Kob82, Lüb83**]. Signalons au passage qu'il existe des méthodes purement algébriques (dues à Miyaoka essentiellement) pour retrouver ce résultat.

Dans le cas singulier, les choses se compliquent sérieusement, car on ne dispose pas vraiment de notion de métrique singulière pour les fibrés vectoriels de rang > 1. Or, c'est exactement ce dont on aurait besoin : on voudrait pouvoir interpréter la métrique de Kähler-Einstein singulière comme métrique hermitienne singulière sur le faisceau tangent $\mathscr{T}_X$, puis généraliser la partie "facile" de la correspondance de Kobayashi-Hitchin dans ce contexte.

Mais à vrai dire, cette stratégie ne marche pas très bien. En revanche, si l'on ne travaille qu'avec des approximations (lisses) de la métrique Kähler-Einstein (disons dans une résolution), alors Enoki [**Eno88**] a montré à la fin des années 80 que l'on peut reproduire les arguments usuels du cas lisse pour montrer la semi-stabilité du tangent dans le cas de singularités *canoniques*, car on a alors des informations assez précises sur le comportement de la métrique KE comme l'a montré Yau [**Yau78b**] – précisons qu'il suffit de travailler avec une classe ample fixée, mais avec un membre de droite dégénéré dans l'équation de Monge-Ampère, cas déjà étudié par Yau.

A l'époque d'Enoki, la théorie de métrique de Kähler-Einstein singulières n'était pas développée, et les outils manquaient pour estimer précisément le comportement des métriques dégénérées. A l'aide de la technologie moderne (sic) nous avons pu généraliser le résultat d'Enoki au cas de variétés à singularités log canoniques, et même obtenir la polystabilité :

Théorème 1.5.3. — *Soit X une variété à singularités log canoniques telle que K_X est ample. Alors $\mathscr{T}_X$ est polystable par rapport à K_X.*

Comme corollaire immédiat de ce résultat, on retrouve que le faisceau tangent d'une telle variété n'a pas de section globale non nulle (car $\mathscr{T}_X$ est de degré < 0), obtenu précédemment par [**BHPS12**] et [**BG13**]. Ou de manière équivalente, une telle variété a un groupe d'automorphisme fini.

Un mot sur la preuve, pour laquelle nous renvoyons au chapitre 5. Pour la semi-stabilité, on suit la stratégie d'Enoki en commençant par le cas de singularités log terminales (le cas lc s'en déduit par approximation, de manière assez surprenante finalement). Bien sûr, il arrive un moment où cela coince, et il faut alors analyser précisément les singularités de la métrique KE associée à une paire log lisse klt (X, D) telle que $K_X + D$ est ample. L'estimée que nous obtenons est à notre connaissance nouvelle, et passe par une généralisation de l'estimée de laplacien obtenue par la formule de Chern-Lu.

Quant à la polystabilité, elle utilise de manière cruciale la convergence $\mathscr{C}^\infty$ sur les compacts de $X_{\rm reg}$ des approximants de la métrique KE (résultat qui donne en particulier la lissité de la métrique KE sur le lieu régulier) prouvée dans [**BG13**], cf aussi le chapitre 4. A part cela, il faut écrire avec précaution les arguments habituels et essayer de les passer à la limite quand la situation dégénère...

La preuve du théorème est assez souple est s'adapte sans trop de difficultés à d'autres contextes ; plus précisément, nous montrons :

Théorème 1.5.4. — *Soit X un espace compact kählerien à singularités log canoniques.*

(*i*) *Si $c_1(K_X) = 0$, alors $\mathscr{T}_X$ est semi-stable par rapport à toute polarisation.*
(*ii*) *Si K_X (resp. $-K_X$) est nef, alors Ω_X (resp. $\mathscr{T}_X$) est génériquement semi-positif.*

Nous renvoyons au chapitre 5 pour la définition précise de semi-positivité générique employée ici, mais essentiellement, cela signifie que la pente de tout quotient cohérent de Ω_X (resp. $\mathscr{T}_X$) est positive. Notons que le deuxième énoncé est une forme faible (mais dans le cas kählerien tout de même) du célèbre théorème de semi-positivité générique de Miyaoka [**Miy87**] affirmant que le faisceau cotangent d'une variété projective normale non uniréglée est génériquement nef (or il est facile de constater qu'une variété à canonique nef n'est pas uniréglée).

1.6. Projets

J'ai actuellement plusieurs projets en tête, certains étant déjà assez avancés.

Tout d'abord, après la rédaction de cette thèse, j'ai écrit avec Mihai Păun un article [**GP13**] où nous étendons les résultats de [**CGP11**] puis des chapitres 2 et 3 de ce manuscrit en s'abstrayant de l'hypothèse récurrente sur les angles des cônes (devant être originellement inférieurs à π). La clé réside dans l'observation selon laquelle même si la courbure de la métrique conique modèle n'est pas bornée inférieurement, elle reste dominée (par dessous) par le dd^c d'une fonction *bornée* rendant l'estimée de laplacien possible.

Ensuite, une question assez naturelle se pose une fois qu'on a compris les singularités de la métrique de Kähler-Einstein associée à une paire log lisse (X, D) avec X lisse, et D un diviseur *effectif* à coefficients plus petits que 1 (au sens large) tel que $K_X + D$ est ample. Tout du moins sous l'hypothèse que les coefficients de D sont plus grands que $1/2$, on sait donc depuis [**Gue12b**] que la métrique KE est à singularités mixtes Poincaré et coniques. En réalité, la première étape consiste à montrer que le potentiel (par rapport à une métrique lisse de référence) de la métrique KE est de la forme $\sum -\log\log^2|s_k|^2 + O(1)$ où la somme porte sur les composantes de D à coefficients 1. Et pour cette étape, on n'a pas besoin de l'hypothèse sur les coefficients.

Que se passe-t-il maintenant si le diviseur D s'écrit $\sum a_i D_i$ avec des coefficients $a_i \leqslant 1$, éventuellement négatifs (et toujours $K_X + D$ ample) ? L'existence de la métrique KE n'est pas complètement évidente déjà, mais on peut utiliser les arguments (un peu massue) de [**BG13**] pour voir que la métrique existe bien, est unique, et lisse hors de D. Si tous les a_i sont strictement plus petits que 1, alors l'estimée de Kołodziej permet de voir que le potentiel est borné. A priori, rajouter des zéros dans le membre de droite de l'équation de Monge-Ampère ne devrait pas faire exploser le potentiel. On peut donc conjecturer qu'on dispose de la même estimée $\varphi_{\mathrm{KE}} = \sum \log\log^2|s_k|^2 + O(1)$ dans le cas général. Quand on essaie de reprendre les techniques habituelles (principe du maximum, ou sur/sous-solutions+principe de comparaison plus généralement), on se heurte à une difficulté nouvelle cependant.

Dans un travail en cours avec Damin Wu, et inspirés par [**Don12**], nous arrivons à surmonter cette difficulté en explicitant le noyau de Green de la métrique Poincaré.

Enfin, une question autant naturelle qu'intéressante est de comprendre plus précisément en quel sens la géométrie Poincaré est une limite des géométries coniques lorsque l'angle tend vers 0. Nous avons donné une justification heuristique dans le cas du disque épointé, mais on aimerait que le phénomène soit global.
Plus précisément, prenons une variété X et un diviseur lisse D tel que $K_X + D$ soit ample. Alors pour $\beta > 0$ assez petit, $K_X + (1-\beta)D$ est ample, et il existe donc une unique métrique KE conique ω_β d'angle $2\pi\beta$ le long de D. De même il existe une unique métrique KE Poincaré ω_P. La question précédente peut se formuler en demandant si la géométrie de (X, ω_β) converge en un certain sens vers celle de (X, ω_P). Alors, en utilisant à la fois la méthode variationnelle et la méthode d'estimées classique, je pense savoir montrer que ω_β converge vers ω_P au sens faible sur X, et que la convergence est $\mathscr{C}^\infty$ sur les compacts de $X \setminus D$.

CHAPITRE 2

KÄHLER-EINSTEIN METRICS WITH MIXED POINCARÉ AND CONE SINGULARITIES

Introduction

Let X be a compact Kähler manifold of dimension n, and $D = \sum a_i D_i$ an effective $\mathbb{R}$-divisor with simple normal crossing support such that the a_i's satisfy the following inequality: $0 < a_i \leqslant 1$. We write $X_0 = X \setminus \mathrm{Supp}(D)$.

Our local model is given by the product $X_{\mathrm{mod}} = (\mathbb{D}^*)^r \times (\mathbb{D}^*)^s \times \mathbb{D}^{n-(s+r)}$ where $\mathbb{D}$ (resp. $\mathbb{D}^*$) is the disc (resp. punctured disc) of radius $1/2$ in $\mathbb{C}$, the divisor being $D_{\mathrm{mod}} = d_1[z_1 = 0] + \cdots + d_r[z_r = 0] + [z_{r+1} = 0] + \cdots + [z_{r+s} = 0]$, with $d_i < 1$. We will say that a metric ω on X_{mod} has mixed Poincaré and cone growth (or singularities) along the divisor D_{mod} if there exists $C > 0$ such that

$$C^{-1}\omega_{\mathrm{mod}} \leqslant \omega \leqslant C\,\omega_{\mathrm{mod}}$$

where

$$\omega_{\mathrm{mod}} := \sum_{j=1}^{r} \frac{idz_j \wedge d\bar{z}_j}{|z_j|^{2d_j}} + \sum_{j=r+1}^{s} \frac{idz_j \wedge d\bar{z}_j}{|z_j|^2 \log^2 |z_j|^2} + \sum_{j=r+s+1}^{n} idz_j \wedge d\bar{z}_j$$

is simply the product metric of the standard cone metric on $(\mathbb{D}^*)^r$, the Poincaré metric on $(\mathbb{D}^*)^s$, and the euclidian metric on $\mathbb{D}^{n-(s+r)}$.

This notion makes sense for global (Kähler) metrics ω on the manifold X_0; indeed, we can require that on each trivializing chart of X where the pair (X, D) becomes $(X_{\mathrm{mod}}, D_{\mathrm{mod}})$ (those charts cover X), ω is equivalent to ω_{mod} just like above, and this does not depend on the chosen chart.

Our goal will then be to find, whenever this is possible, Kähler metrics on X_0 having constant Ricci curvature and mixed Poincaré and cone growth along D. Those metrics will naturally be called Kähler-Einstein metrics. For reasons which will appear in section 2.1.2 and more precisely in Remark 2.1.3, we will restrict ourselves to looking for Kähler-Einstein metrics with negative curvature.

The existence of Kähler-Einstein metrics (in the previously specified sense) has already been studied in various contexts and for multiple motivations. The logarithmic case (all coefficients of D are equal to 1) has been solved when $K_X + D$ is assumed to be ample by R. Kobayashi [**Kob84**] and G.Tian-S.T.Yau [**TY87**], the latter considering also orbifold coefficients for the fractional part $D_{klt} = \sum_{\{a_i<1\}} a_i D_i$ of D, that is of the form $1 - \frac{1}{m}$ for some integers $m > 1$. Our main result extends this when the coefficients of D_{klt} are no longer orbifold coefficients, but are any real numbers $a_i \geqslant 1/2$ (condition which is realized if a_i is of orbifold type):

Theorem A. — *Let X be a compact Kähler manifold and $D = \sum a_i D_i$ a $\mathbb{R}$-divisor with simple normal crossing support such that $K_X + D$ is ample. We assume furthermore that the coefficients of D satisfy the following inequalities:*

$$1/2 \leqslant a_i \leqslant 1.$$

Then $X \setminus \mathrm{Supp}(D)$ carries a unique Kähler-Einstein metric ω_{KE} with curvature -1 having mixed Poincaré and cone singularities along D.

The conic case, *ie* when the coefficients of D are stricly less than 1), under the assumption that $K_X + D$ is positive or zero, has been studied by R. Mazzeo [**Maz99**], T. Jeffres [**Jef00**] and recently resolved independently by S. Brendle [**Bre11**] and R. Mazzeo, T. Jeffres, Y. Rubinstein [**JMR11**] in the case of an (irreducible) smooth divisor, and by [**CGP11**] in the general case of a simple normal crossing divisor (having though all its coefficients greater than $\frac{1}{2}$). In the conic case where $K_X + D < 0$, some interesting existence results were obtained by R. Berman in [**Ber11**] and T. Jeffres, R. Mazzeo and Y. Rubinstein in [**JMR11**]. Let us finally mention that in [**JMR11**], it is proved that the potential of the Kähler-Einstein metric has polyhomogeneous expansion, which is much stronger than the assertion on the cone singularities of this metric.

Let us now give a sketch of the proof by detailing the organization of the paper.

The first step is, as usual, to relate the existence of Kähler-Einstein metrics to some particular Monge-Ampère equations. We explain this link in Proposition 2.2.5. The idea is that any negatively curved Kähler-Einstein metric on X_0 with appropriate boundary conditions extends to a Kähler current of finite energy in $c_1(K_X + D)$ satisfying on X a Monge-Ampère equation of the type $\omega_\varphi^n = e^{\varphi - \varphi_D} \omega^n$ where ω is a Kähler form on X, and $\varphi_D = \sum a_i \log |s_i|^2 + (\text{smooth terms})$. One may observe that as soon as some a_i equals 1, the measure $e^{-\varphi_D}\omega^n$ has infinite mass.

The uniqueness of the solution metric will then follow from the so-called comparison principle established by V.Guedj and A.Zeriahi in [**GZ07**] for this class of finite energy currents.

We are then reduced to solving some singular Monge-Ampère equation. The strategy consists in working on the open manifold $X_{lc} := X \setminus D_{lc}$, and we are led to the following equation: $\omega_\varphi^n = e^{\varphi - \varphi_{D_{klt}}} \omega^n$ where this time ω is a Kähler form on X_{lc} with Poincaré singularities along D_{lc}, and $\varphi_{D_{klt}} = \sum_{\{a_i<1\}} a_i \log |s_i|^2 + (\text{smooth terms})$. If $\varphi_{D_{klt}}$ were smooth, one could simply apply the results of Kobayashi and Tian-Yau. As it is not the case, we adapt the strategy of Campana-Guenancia-Păun to this setting:

We start in section 2.4.1 by regularizing $\varphi_{D_{klt}}$ into a smooth function (on X_{lc}) $\varphi_{D_{klt},\varepsilon}$ and introducing smooth approximations ω_ε of the cone metric on X_{lc} having Poincaré singularities along D_{lc}. Then we consider the regularized equation $\omega_{\varphi_\varepsilon}^n = e^{\varphi_\varepsilon - \varphi_{D_{klt},\varepsilon}}\omega_\varepsilon^n$ which we can solve for every $\varepsilon > 0$ (we are in the logarithmic case). The point is to construct our desired solution φ as the limit of $(\varphi_\varepsilon)_\varepsilon$; this is made possible by controlling (among other things) the curvature of ω_ε, and applying appropriate *a priori* laplacian estimates which we briefly explain in section 2.1.4. The final step is standard: it consists in invoking Evans-Krylov $\mathscr{C}^{2,\alpha}$ interior estimates, and concluding that φ is smooth on X_0 using Schauder estimates.

In the last part of the paper, and as in [**CGP11**], we try to use the Kähler-Einstein metric constructed in the previous sections to obtain the vanishing of some particular holomorphic tensors attached to a pair (X, D), D being still a $\mathbb{R}$-divisor with simple normal crossing support and having coefficients in $[0, 1]$. This specific class consists in the holomorphic tensors which are the global sections of the locally free sheaf $T_s^r(X|D)$ introduced by Campana in [**Cam11b**]: they are holomorphic tensors with prescribed zeros or poles along D. Thanks to their realization as bounded tensors with respect to some (or equivalently, any) twisted metric g with mixed cone and Poincaré singularities along D (cf. Proposition 2.5.3), we can use Theorem A to prove the following:

Theorem B. — *Let (X, D) be a pair satisfying the assumptions of Theorem A. Then, there is no non-zero holomorphic tensor of type (r, s) whenever $r \geqslant s + 1$:*

$$H^0(X, T_s^r(X|D)) = 0.$$

The proof of this results follows closely the one of its analogue in [**CGP11**]: we use a Bochner formula applied to the truncated holomorphic tensors, and the key point is to control the error term. However, a new difficulty pops up here, namely we have to deal with an additional term coming from the curvature of the line bundle $\mathcal{O}_X(\lfloor D \rfloor)$; fortunately, it has the right sign.

2.1. Preliminaries

In this first section devoted to the preliminaries, we intend to fix the notations and the scope of this paper. We also recall some useful objects introduced in [**Kob84**] and [**TY87**] within the framework of the logarithmic case; finally, we explain briefly some a priori estimates which are going to be essential tools in the proof of the main theorem.

2.1.1. Notations and definitions. — All along this work, X will be a compact Kähler manifold of complex dimension n. We will consider effective $\mathbb{R}$-divisors $D = \sum a_i D_i$ with simple normal crossing support, and such that their coefficients a_i belong to $[0, 1]$.

It will be practical to separate the hypersurfaces D_i appearing with coefficient 1 in D from the other ones. For this, we write:

$$\begin{aligned} D &= \sum_{\{a_i<1\}} a_i D_i + \sum_{\{a_i=1\}} D_i \\ &= D_{klt} + D_{lc} \end{aligned}$$

These notations come from the framework of the pairs in birational geometry; klt stands for *Kawamata log-terminal* whereas lc means *log-canonical*. In this language, (X, D) is called a log-smooth lc pair, and (X, D_{klt}) is a log-smooth klt pair. Apart from these practical notations, we will not use this terminology.

We will denote by s_i a section of $\mathcal{O}_X(D_i)$ whose zero locus is the (smooth) hypersurface D_i, and, omitting the dependance in the metric, we write $\Theta(D_i)$ the curvature form of $(\mathcal{O}_X(D_i), h_i)$ for some hermitian metric on $\mathcal{O}_X(D_i)$. Up to scaling the h_i's, one can assume that $|s_i| \leqslant e^{-1}$, and we will make this assumption all along the paper. Finally, we set $X_0 := X \setminus \mathrm{Supp}(D)$ and $X_{lc} := X \setminus \mathrm{Supp}(D_{lc})$.

In the introduction, we mentioned a natural class of growth of Kähler metrics near the divisor D which we called metrics with mixed Poincaré and cone singularities along D. They are the Kähler metrics locally equivalent to the model metric $\omega_{\mathrm{mod}} = \sum_{j=1}^{r} \frac{idz_j\wedge d\bar{z}_j}{|z_j|^{2d_j}} + \sum_{j=r+1}^{s} \frac{idz_j\wedge d\bar{z}_j}{|z_j|^2 \log^2 |z_j|^2} + \sum_{j=r+s+1}^{n} idz_j \wedge d\bar{z}_j$ whenever the pair (X, D) is locally isomorphic to $(X_{\mathrm{mod}}, D_{\mathrm{mod}})$ with $X_{\mathrm{mod}} = (\mathbb{D}^*)^r \times (\mathbb{D}^*)^s \times \mathbb{D}^{n-(s+r)}$ and $D_{\mathrm{mod}} = d_1[z_1 = 0] + \cdots + d_r[z_r = 0] + [z_{r+1} = 0] + \cdots + [z_{r+s} = 0]$, with $d_i < 1$.
The following elementary lemma ensures that given a pair (X, D) as above, Kähler metrics with mixed Poincaré and cone singularities along D always exist:

Lemma 2.1.1. — *The following* $(1, 1)$*-form*

$$\omega_D := \omega_0 + \sum_{\{a_i<1\}} dd^c |s_i|^{2(1-a_i)} - \sum_{\{a_i=1\}} dd^c \log\log \frac{1}{|s_i|^2}$$

defines a Kähler form on X_0 *as soon as* ω_0 *is a sufficiently positive Kähler metric on* X*. Moreover, it has mixed Poincaré and cone singularities along* D*.*

Proof. — This can be seen by a simple computation: combine e.g. [**Cla08**, Proposition 2.1] with [**CG72**, Proposition 2.1] or [**Gri76**, Proposition 2.17]. □

Before we end this paragraph, we would like to emphasize the different role played by the D_i's whether they appear in D with coefficient 1 or stricly less than 1. Here is some explanation: let $0 < \alpha < 1$ be a real number, and $\omega_\alpha = \frac{(1-\alpha)^2 idz\wedge d\bar{z}}{|z|^{2\alpha}(1-|z|^{2(1-\alpha)})^2}$; its curvature is constant equal to -1 on the punctured disc $\mathbb{D}^*$, and it has a cone singularity along the divisor $\alpha[z = 0]$. Then, when α goes to 1, ω_α converges pointwise to the Poincaré metric $\omega_P = \frac{idz\wedge d\bar{z}}{|z|^2 \log^2 |z|^2}$.

In the following, any pair (X, D) *will be implicitely assumed to be composed of a compact Kähler manifold* X *and a* $\mathbb{R}$*-divisor* D *on* X *having simple normal crossing support and* *coefficients belonging to* $[0,1]$.

2.1.2. Kähler-Einstein metrics for pairs. — As explained in the introduction, the goal of this paper is to find a Kähler metric on X_0 with constant Ricci curvature, and having mixed Poincaré and cone singularities along the given divisor D. The second condition is essential and as important as the first one; the proof of the vanishing theorem for holomorphic tensors in the last section will render an account of this and shall surely convince the reader. Let us state properly the definition:

Definition 2.1.2. — A Kähler-Einstein metric for a pair (X,D) is defined to be a Kähler metric ω on X_0 satisfying the following properties:

- $\mathrm{Ric}\,\omega = \mu\,\omega$ for some real number μ;
- ω has mixed Poincaré and cone singularities along D.

Remark 2.1.3. — Unlike cone singularities, Poincaré singularities are intrinsically related to negative curvature geometry:

- The Bonnet-Myers Theorem tells us that in the case where $D_{klt}=0$ (so that we work with complete metrics), there cannot exist Kähler-Einstein metrics in the previous sense with $\mu>0$. However, if $D_{lc}=0$, there may exist Kähler-Einstein metrics with positive curvature, and the question of their existence is often a difficult question (see e.g. [**BBE$^+$11**] or [**Ber11**]).
- As for the Ricci-flat case ($\mu=0$), it also has to be excluded. Indeed, there cannot be any Ricci-flat metric on the punctured disc $\mathbb{D}^*$ with Poincaré singularity at 0; to see this, we write $\omega=\frac{i}{2}e^{2u}dz\wedge d\bar{z}$ such a metric, and then u has to satisfy the following properties: u is harmonic on $\mathbb{D}^*$ and e^{2u} behaves like $\frac{1}{|z|^2\log^2|z|^2}$ near 0, up to constants. But it is well-known that any harmonic function u on D^* can be written $u=\mathrm{Re}(f)+c\log|z|$ for some holomorphic function f on $\mathbb{D}^*$ and some constant $c\in\mathbb{R}$. Clearly, f cannot have an essential singularity at 0; moreover, because of the logarithmic term in the Poincaré metric, f can neither be bounded, nor have a pole at 0. This ends to show that in general (and for local reasons), there does not exist Ricci-flat Kähler-Einstein metric in the sense of the previous definition (whenever $D_{lc}\neq 0$).

For these reasons, we will focus in the following on the case of negative curvature, which we will normalize by taking $\mu=-1$.

2.1.3. The logarithmic case. — For the sake of completeness, we will briefly recall in this section the proof of the main result (Theorem 2.3.1) in the logarithmic case, namely when $D=D_{lc}$, *ie* when $D_{klt}=0$. As we already explained, this was achieved by Kobayashi [**Kob84**] and Tian-Yau [**TY87**] in a very similar way. In this section, we will assume that (X,D) is logarithmic, so that $X_0=X_{lc}$.

We will use the following terminology which is convenient for the following:

Definition 2.1.4. — We say that a Kähler metric ω on X_0 is of Carlson-Griffiths type if there exists a Kähler form ω_0 on X such that $\omega=\omega_0-\sum_K dd^c\log\log\frac{1}{|s_k|^2}$.

As observed in Lemma 2.1.1, such a metric always exists, and it has Poincaré singularities along D. In [**CG72**], Carlson and Griffiths introduced such a metric for some $\omega_0 \in c_1(K_X + D)$. The reason why we exhibit this particular class of Kähler metric on X_0 having Poincaré singularities along D is that we have an exact knowledge on its behaviour along D, much more precise that its membership of the previously cited class. For example, Lemma 2.1.6 mirrors this fact.

We start from a compact Kähler manifold X with a simple normal crossing divisor $D = \sum D_k$ such that $K_X + D$ is ample. We want to find a Kähler metric ω_{KE} on $X_0 = X \setminus D$ with $-\mathrm{Ric}\,\omega_{\mathrm{KE}} = \omega_{\mathrm{KE}}$, and having Poincaré singularities along D. If we temporarily forget the boundary condition, the problem amounts to solve the following Monge-Ampère equation on X_0:

$$(\omega + dd^c\varphi)^n = e^{\varphi+F}\omega^n$$

where ω is a Kähler metric on X_0 of Carlson-Griffiths type (cf. Definition 2.1.4), and $F = -\log\left(\prod |s_k|^2 \log^2 |s_k|^2 \cdot \omega^n/\omega_0^n\right) + (\mathrm{smooth\,terms\,on}\,X)$ for some Kähler metric ω_0 on X.

The key point is that (X_0, ω) has bounded geometry at any order. Let us get a bit more into the details. To simplify the notations, we will assume that D is irreducible, so that locally near a point of D, X_0 is biholomorphic to $\mathbb{D}^* \times \mathbb{D}^{n-1}$, where $\mathbb{D}$ (resp. $\mathbb{D}^*$) is the unit disc (resp. punctured disc) of $\mathbb{C}$. We want to show that, roughly speaking, the components of ω in some appropriate coordinates have bounded derivatives at any order. The right way to formalize it consists in introducing quasi-coordinates: they are maps from an open subset $V \subset \mathbb{C}^n$ to X_0 having maximal rank everywhere. So they are just locally invertible, but these maps are not injective in general.
To construct such quasi-coordinates on X_0, we start from the univeral covering map $\pi : \mathbb{D} \to \mathbb{D}^*$, given by $\pi(w) = e^{\frac{w+1}{w-1}}$. Formally, it sends 1 to 0. The idea is to restrict π to some fixed ball $B(0, R)$ with $1/2 < R < 1$, and compose it (at the source) with a biholomorphism Φ_η of $\mathbb{D}$ sending 0 to η, where η is a real parameter which we will take close to 1. If one wants to write a formula, we set $\Phi_\eta(w) = \frac{w+\eta}{1+\eta w}$, so that the quasi-coordinate maps are given by $\Psi_\eta = \pi \circ \Phi_\eta \times \mathrm{Id}_{\mathbb{D}^{n-1}} : V = B(0, R) \times \mathbb{D}^{n-1} \to \mathbb{D}^*$, with $\Psi_\eta(v, v_2, \ldots, v_n) = (e^{\frac{1+\eta}{1-\eta}\frac{v+1}{v-1}}, v_2, \ldots, v_n)$.
Once we have said this, it is easy to see that X_0 is covered by the images $\Psi_\eta(V)$ when η goes to 1, and for all the trivializing charts for X, which are in finite number. Now, an easy computation shows that the derivatives of the components of ω with respect to the v_i's are bounded uniformly in η. This can be thought as a consequence of the fact that the Poincaré metric is invariant by any biholomorphism of the disc.

At this point, it is natural to introduce the Hölder space of $\mathscr{C}^{k,\alpha}_{qc}$-functions on X_0 using the previously introduced quasi-coordinates:

Definition 2.1.5. — For a non-negative integer k, a real number $\alpha \in]0.1[$, we define:

$$\mathscr{C}^{k,\alpha}_{qc}(X_0) = \{u \in \mathscr{C}^k(X_0);\ \sup_{V,\eta} ||u \circ \Psi_\eta||_{k,\alpha} < +\infty\}$$

where the supremum is taken over all our quasi-coordinate maps V (which cover X_0). Here $||\cdot||_{k,\alpha}$ denotes the standard $\mathscr{C}^{k,\alpha}_{qc}$-norm for functions defined on a open subset of $\mathbb{C}^n$.

The following fact, though easy, is very important for our matter:

Lemma 2.1.6. — *Let ω be a Carlson-Griffiths type metric on X_0, and ω_0 some Kähler metric on X. Then*

$$F_0 := \log\left(\prod |s_k|^2 \log^2 |s_k|^2 \cdot \omega^n/\omega_0^n\right)$$

belongs to the space $\mathscr{C}^{k,\alpha}_{qc}(X_0)$ for every k and α.

Proof. — The first remark is that F_0 is bounded (cf. [**Kob84**, Lemma 1.(ii)] or the beginning of section 2.4.2.3), and $F_0 \in \mathscr{C}^{k,\alpha}_{qc}(X_0)$ if and only if $e^{F_0} \in \mathscr{C}^{k,\alpha}_{qc}(X_0)$. So in the following, we will deal with e^{F_0}.
Then, as the (elementary) computations of Lemma 2.4.3 show, it is enough to check that the functions on $\mathbb{D}^*$ (say with radius 1/2) defined by $z \mapsto \frac{1}{\log |z|^2}$, $z \mapsto |z|^2 \log |z|^2$ and $z \mapsto |z|^2 \log^2 |z|^2$ are in $\mathscr{C}^{k,\alpha}_{qc}(\mathbb{D}^*)$. But in the quasi-coordinates given by Φ_η, $\frac{1}{\log |z|^2} = \frac{1}{2}\cdot \frac{1-\eta}{1+\eta}\frac{|v|^2-1}{|v-1|^2}$ and $|z|^2 \log^\alpha |z|^2 = \left(\frac{1}{2}\cdot\frac{1+\eta}{1-\eta}\frac{|v-1|^2}{|v|^2-1}\right)^\alpha e^{2\cdot\frac{1+\eta}{1-\eta}\frac{|v|^2-1}{|v-1|^2}}$, for $v \in B(0,R)$ with $R<1$, and where $\alpha \in \mathbb{R}$. Now there is no difficulty in seeing that these two functions of v are bounded when η goes to 1 (actually this property does not depend on the chosen coordinates), and so are their derivatives (still with respect to v); this is obvious for the first function, and for the second one, it relies on the fact that $x^m e^{-x}$ goes to 0 as $x \to +\infty$, for all $m \in \mathbb{Z}$. □

The end of the proof consists in showing that the Monge-Ampère equation $(\omega + dd^c\varphi)^n = e^{\varphi+f}\omega^n$ has a unique solution $\varphi \in \mathscr{C}^{k,\alpha}_{qc}(X_0)$ for all functions $f \in \mathscr{C}^{k,\alpha}_{qc}(X_0)$ with $k \geqslant 3$. This can be done using the continuity method in the quasi-coordinates. In particular, applying this to $f = F := -\log\left(\prod |s_k|^2 \log^2 |s_k|^2 \cdot \omega^n/\omega_0^n\right) + (\text{smooth terms on } X)$ (cf beginning of the section), which the previous lemma allows to do, this will prove the existence of a negatively curved Kähler-Einstein metric, which is equivalent to ω (in the *strong* sense: $\varphi \in \mathscr{C}^{k,\alpha}_{qc}(X_0)$ for all k, α).

To summarize, the theorem of Kobayashi and Tian-Yau is the following:

Theorem 2.1.7 **([Kob84, TY87])**. — *Let (X, D) be a logarithmic pair, ω a Kähler form of Carlson-Griffiths type on X_0, and $F \in \mathscr{C}^{k,\alpha}_{qc}(X_0)$ for some $k \geqslant 3$. Then there exists $\varphi \in \mathscr{C}^{k,\alpha}_{qc}(X_0)$ solution to the following equation:*

$$(\omega + dd^c\varphi)^n = e^{\varphi+F}\omega^n$$

In particular if $K_X + D$ is ample, then there exists a (unique) Kähler-Einstein metric of curvature -1 equivalent to ω.

2.1.4. *A priori* estimates. — In this section, we recall the classical estimates valid for a large class of complete Kähler manifolds; they are derived from the classical estimates over compact manifolds using the generalized maximum principle of Yau [**Yau78a**]. We will use them in an essential manner in the course of the proof of our main theorem. Indeed, our proof is based upon a regularization process, and in order to guarantee the existence of the limiting object, we need to have a control on the $\mathscr{C}^k$ norms.

Theorem 2.1.8. — *Let X be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a $\mathscr{C}^2$ function which is bounded from below on M. Then for every $\varepsilon > 0$, there exists $x \in X$ such that at x,*

$$|\nabla f| < \varepsilon, \quad \Delta f > -\varepsilon, \quad f(x) < \inf_X f + \varepsilon.$$

From this, we easily deduce the following result, stated in [**CY80**, Proposition 4.1].

Proposition 2.1.9. — *Let (X, ω) be a n-dimensional complete Kähler manifold, and $F \in \mathscr{C}^2(X)$ a bounded function. We assume that we are given $u \in \mathscr{C}^2(X)$ satisfying $\omega + dd^c u > 0$ and*

$$(\omega + dd^c u)^n = e^{u+F}\omega^n$$

Suppose that the bisectional curvature of (X, ω) is bounded below by some constant, and that u is a bounded function. Then

$$\sup_X |u| \leqslant \sup_X |F|.$$

We emphasize the fact that the previous estimate does not depend on the lower bound for the bisectional curvature of (X, ω).
As for the Laplacian estimate, we have the following (we could also have used [**CY80**, Proposition 4.2]):

Proposition 2.1.10. — *Suppose that the bisectional curvature of (X, ω) is bounded below by some constant $-B, B > 0$, and that u as well as its Laplacian Du are bounded functions on X. If $\omega + dd^c u$ defines a complete Kähler metric on X with Ricci curvature bounded from below, then*

$$\sup_X (n + Du) \leqslant C$$

where $C > 0$ only depends on $\sup |F|$, $\inf DF$, *B and n.*

Sketch of the proof. — We set $\omega' = \omega + dd^c u$, and D' is defined to be the Laplacian with respect to ω'.
Using [**CGP11**, Lemma 2.2], we obtain $D'(\mathrm{tr}_\omega \omega') \geqslant \frac{DF}{\mathrm{tr}_{\omega'}\omega} - B\mathrm{tr}_{\omega'}\omega$, and from this we may deduce that

$$D'(\mathrm{tr}_\omega \omega' - (C_1 + 1)u) \geqslant \mathrm{tr}_{\omega'}\omega - C_2$$

where C_1, C_2 are constant depending only B, $\inf DF$ and n. The assumptions allow us to use the generalized maximum principle stated as Theorem 2.1.8 to show that $\sup \mathrm{tr}_{\omega'}\omega \leqslant C_3$. As $\omega' = e^{F+u}\omega$, and as we have at our disposal uniform estimates on $\sup |u|$ thanks to 2.1.9, the usual arguments work here to give a uniform bound $\sup (n + Du) \leqslant C$. We refer e.g. to [**CGP11**, section 2] for more details. □

2.2. Uniqueness of the Kähler-Einstein metric

In this section, we begin to investigate the questions raised in the introduction concerning the existence of Kähler-Einstein metrics for pairs (X, D). The first thing to do is, as usual, to relate the existence of theses metrics to the existence of solutions for some Monge-Ampère equations. We will be in a singular case, so we have to specify the class of ω-psh functions to which we are going to apply the Monge-Ampère operator. This is the aim of the few following lines, where we will recall some recent (but relatively basic) results of pluripotential theory. We refer to [**GZ07**] or [**BEGZ10**] for a detailed treatment.

2.2.1. Energy classes for quasi-psh functions. — Let ω be a Kähler metric on X; the class $\mathcal{E}(X,\omega)$ is defined to be composed of ω-psh functions φ such that their non-pluripolar Monge-Ampère $(\omega + dd^c\varphi)^n$ has full mass $\int_X \omega^n$ (cf. [**GZ07**], [**BEGZ10**]). An alternate way to apprehend those functions is to see them as the largest class where one can define $(\omega + dd^c\varphi)^n$ as a measure which does not charge pluripolar sets. Those functions satisfy the so-called comparison principle, which we are going to use in an essential manner for the uniqueness of our Kähler-Einstein metric:

Proposition 2.2.1 **(Comparison Principle, [GZ07]).** — *Let $\varphi, \psi \in \mathcal{E}(X,\omega)$. Then we have:*

$$\int_{\{\varphi<\psi\}} (\omega + dd^c\psi)^n \leqslant \int_{\{\varphi<\psi\}} (\omega + dd^c\varphi)^n.$$

An important subset of $\mathcal{E}(X,\omega)$ is the class $\mathcal{E}^1(X,\omega)$ of functions in the class $\mathcal{E}(X,\omega)$ having finite $\mathcal{E}^1$-energy, namely $\mathcal{E}^1(\varphi) := \int_X |\varphi|(\omega + dd^c\varphi)^n < +\infty$. Every smooth (or even bounded) ω-psh function belongs to this class.
In order to state an useful result for us, we recall the notion of *capacity* attached to a compact Kähler manifold (X,ω), as introduced in [**GZ05**], generalizing the usual capacity of Bedford-Taylor ([**BT82**]): for every Borel subset K of X, we set:

$$\mathrm{Cap}_\omega(K) := \sup\left\{\int_K \omega_\varphi^n;\ \varphi \in \mathrm{PSH}(X,\omega),\ 0 \leqslant \varphi \leqslant 1\right\}$$

There is an useful criteria to show that some ω-psh function belongs to the class $\mathcal{E}^1(X,\omega)$ without checking that it has full Monge-Ampère mass, but only using the capacity decay of the sublevel sets. It appears in different papers, among which [**GZ07**, Lemma 5.1], [**BGZ08**, Proposition 2.2], [**BBGZ09**, Lemma 2.9]:

Lemma 2.2.2. — *Let $\varphi \in \mathrm{PSH}(X,\omega)$. If*

$$\int_{t=0}^{+\infty} t^n \,\mathrm{Cap}_\omega\{\varphi < -t\}\, dt < +\infty$$

then $\varphi \in \mathcal{E}^1(X,\omega)$.

Now we have enough background about these objects to state and prove the result we will use in the next section. Let us first fix the notations.
Let (X,ω_0) be a Kähler manifold, and $D = \sum_{k\in K} D_k$ a simple normal crossing divisor. We choose sections s_k of $\mathcal{O}_X(D_k)$ whose divisor is precisely D_k, and we fix some smooth hermitian metrics on those line bundles. We can assume that $|s_k| \leqslant e^{-1}$, and we know that, up to scaling the metrics, one may assume that $\omega_0 - \sum_k dd^c \log\log \frac{1}{|s_k|^2}$ is positive on X_0, and defines a Kähler current on X.

Proposition 2.2.3. — *The function*

$$\varphi_0 = -\sum_{k\in K} \log\log \frac{1}{|s_k|^2}$$

belongs to the class $\mathcal{E}^1(X,\omega_0)$.

Proof. — We want to apply Lemma 2.2.2. To compute the global capacity as defined above, or at least know the capacity decay of the sublevel sets, it is convenient to use the Bedford-Taylor capacity. But a result due to Kołodziej [**Koł01**] (see also [**GZ05**, Proposition 2.10]), states that up to universal multiplicative constants, the capacity can be computed by the local Bedford-Taylor capacities on the trivializing charts of X.

Therefore, we are led to bound from above $\mathrm{Cap}_{BT}\{u<-t\}$ in the unit polydisc of $\mathbb{C}^n$, where $u=\sum_{i=1}^p -\log(-\log|z_i|^2)$ for some $p\leqslant n$. As

$$\{u<-t\}\subset\bigcup_{i=1}^p\left\{-\log(-\log|z_i|^2)<-\frac{t}{p}\right\}$$

one can now assume that $p=1$. But $\mathrm{Cap}_{BT}\{\log|z|^2<-t)=2/t$ (see e.g [**Dem**, Example 13.10]), whence $\mathrm{Cap}_{BT}\{-\log(-\log|z_i|^2)<-t)=2e^{-t}$. The result follows. □

Remark 2.2.4. — An alternate way to proceed is to show that the smooth approximations $\varphi_\varepsilon:=-\sum_{k\in K}\log\log\frac{1}{|s_k|^2+\varepsilon^2}$ of φ_0 have (uniformly) bounded $\mathcal{E}^1$-energy, which also allows to conclude that $\varphi_0\in\mathcal{E}^1(X,\omega_0)$ thanks to [**BEGZ10**, Proposition 2.10 & 2.11].

2.2.2. From Kähler-Einstein metrics to Monge-Ampère equations. — The following proposition explains how to relate Kähler-Einstein metrics for a pair (X,D) and some Monge-Ampère equations, the difficulty being here that we have to deal with singular weights/potentials for which the definitions and properties of the Monge-Ampère operators are more complicated than in the smooth case. Note that this result generalizes [**Ber11**, Proposition 5.1]:

Proposition 2.2.5. — *Let* X *be a compact Kähler manifold, and* $D=\sum a_jD_j$ *an effective* $\mathbb{R}$*-divisor with simple normal crossing support, such that* $a_j\leqslant 1$ *for all* j*. We assume that* K_X+D *is ample, and we choose a Kähler metric* $\omega_0\in c_1(K_X+D)$*. Then any Kähler metric* ω *on* X_0 *satisfying:*

- $-\mathrm{Ric}\,\omega=\omega$ *on* X_0*;*
- *There exists* $C>0$ *such that:*

$$C^{-1}\omega^n\leqslant\frac{\omega_0^n}{\prod_{\{a_i<1\}}|s_i|^{2a_i}\prod_{\{a_i=1\}}|s_i|^2\log^2|s_i|^2}\leqslant C\omega^n$$

extends to a Kähler current $\omega=\omega_0+dd^c\varphi$ *on* X *where* $\varphi\in\mathcal{E}^1(X,\omega_0)$ *is a solution of*

$$(\omega_0+dd^c\varphi)^n=e^{\varphi-\varphi_D}\omega_0^n$$

and $\varphi_D=\sum_{r\in J\cup K}a_r\log|s_r|^2+f$ *for some* $f\in\mathscr{C}^\infty(X)$*. Furthermore there exists at most one such metric* ω *on* X_0*.*

Remark 2.2.6. — One can observe that although $e^{\varphi-\varphi_D}\omega_0^n$ has finite mass, $e^{-\varphi_D}\omega_0^n$ does not (as soon as $D_{lc}\neq 0$).

Proof. — We recall that $\Theta(D_i)$ denotes the curvature of $(\mathcal{O}_X(D_i), h_i)$, and we write $\Theta(D_{klt}) = \sum_{\{a_i<1\}} a_i\Theta(D_i)$, $\Theta(D_{lc}) = \sum_{\{a_i=1\}} \Theta(D_i)$ and $\Theta(D) = \Theta(D_{klt}) + \Theta(D_{lc})$. All those forms are smooth on X.
Let us define a smooth function ψ on X_0 by:

$$\psi_0 := \log\left(\frac{\prod_{j\in J}|s_j|^{2a_j}\prod_{k\in K}|s_k|^2\log^2|s_k|^2\,\omega^n}{\omega_0^n}\right)$$

By assumption, ψ_0 is bounded on X_0, so that $\psi := \psi_0 - \sum_k \log\log^2\frac{1}{|s_k|^2}$ is bounded above on X_0. On this set, we have

$$dd^c\psi = \omega + \mathrm{Ric}\,\omega_0^n + \Theta(D)$$

so that ψ is $M\omega_0$-psh for some $M>0$ big enough. As it is bounded above, it extends to a (unique) $M\omega_0$-psh function on the whole X, which we will also denote by ψ. Let now f be a smooth potential on X of $\mathrm{Ric}\,\omega_0^n + \omega_0 - \Theta(D)$. It is easily shown that $\varphi := \psi - f$ satisfies $\omega_0 + dd^c\varphi = \omega$ on X_0.

From the definition of φ, we see that $\varphi = 2\varphi_0 + \mathcal{O}(1)$, where $\varphi_0 = -\sum_{k\in K}\log\log\frac{1}{|s_k|^2}$. Therefore, Proposition 2.2.3 ensures that $\varphi \in \mathcal{E}^1(X,\omega_0)$, so that its non-pluripolar Monge-Ampère $(\omega_0 + dd^c\varphi)^n$ satisfies the equation

$$\begin{aligned}(\omega_0 + dd^c\varphi)^n &= \frac{e^{\varphi - f}\omega_0^n}{\prod_{r\in J\cup K}|s_r|^{2a_r}}\\ &= e^{\varphi-\varphi_\Delta}\omega_0^n\end{aligned}$$

on the whole X, with the notations of the statement. By the comparison principle (Proposition 2.2.1), if the previous equation had two solutions $\varphi, \psi \in \mathcal{E}^1(X,\omega_0)$, then on the set $A = \{\varphi < \psi\}$, we would have

$$\int_A e^{\psi-\varphi_\Delta}\omega_0^n \leqslant \int_A e^{\varphi-\varphi_\Delta}\omega_0^n$$

but on A, $e^\psi > e^\varphi$ so that A has zero measure with repect to the measure $e^{-\varphi_\Delta}\omega_0^n$, so it has zero measure with respect to ω_0^n. We can do the same for $B = \{\psi < \varphi\}$, so that $\{\varphi = \psi\}$ has full measure with respect to ω_0^n. As φ, ψ are ω_0-psh, they are determined by their data almost everywhere, so they are equal on X. This finishes to conclude that our φ is unique, so that the proposition is proved. □

Remark 2.2.7. — In the logarithmic case ($D = D_{lc}$), the metrics at stake are complete, so that their uniqueness follow from the generalized maximum principle of Yau (cf. [**Kob84**], [**TY87**] e.g). In the conic case, Kołodziej's theorem [**Koł98**] ensures that the potentials we are dealing with are continuous, and the unicity follows from the classical comparison principle established in [**BT82**, Theorem 4.1].

As Kähler metrics with mixed Poincaré and cone singularities clearly satisfy the second condition of the proposition, we deduce that any negatively curved normalized Kähler-Einstein metric must be obtained by solving the global equation $(\omega_0 + dd^c\varphi)^n = e^{\varphi-\varphi_D}\omega_0^n$ on X, for $\varphi \in \mathcal{E}^1(X,\omega_0)$, and $\varphi_D = \sum_{r\in J\cup K} a_r \log|s_r|^2 + f$ for some $f \in \mathscr{C}^\infty(X)$. We will now show how to solve the previous equation, and derive from this the existence of negatively curved Kähler-Einstein metrics and their zero-th order asymptotic along D.

2.3. Statement of the main result

Here is a result which encompasses the previous results of [**CGP11**], Kobayashi ([**Kob84**]) and Tian-Yau ([**TY87**]). This provides a (positive) partial answer to a question raised in [**CGP11**, section 10].

Theorem 2.3.1. — *Let X be a compact Kähler manifold, and $D=\sum a_i D_i$ an effective $\mathbb{R}$-divisor with simple normal crossing support such that its coefficients satisfy the inequalities:*

$$1/2 \leqslant a_i \leqslant 1.$$

Then for any Kähler form ω on X_{lc} of Carlson-Griffiths type and any function $f \in \mathscr{C}^{k,\alpha}_{qc}(X_{lc})$ with $k \geqslant 3$, there exists a Kähler metric $\omega_\infty = \omega + dd^c\varphi$ on X_0 solution to the following equation:

$$(\omega + dd^c\varphi)^n = \frac{e^{\varphi+f}}{\prod_{\{a_i<1\}} |s_i|^{2a_i}}\, \omega^n$$

such that ω_∞ has mixed Poincaré and cone singularities along D.

We refer to section 2.1.3 and more precisely to Definition 2.1.5 for the definition of the space $\mathscr{C}^{k,\alpha}_{qc}(X_{lc})$; one important class of functions belonging to $\mathscr{C}^{k,\alpha}_{qc}(X_{lc})$ is pointed out in Lemma 2.1.6, and we will use it for proving the following result.

Corollary 2.3.2. — *Let (X,D) be a pair such that $D=\sum a_i D_i$ is a divisor with simple normal crossing support whose coefficients satisfy the inequalities*

$$1/2 \leqslant a_i \leqslant 1.$$

If $K_X + D$ is ample, then X_0 carries a unique Kähler-Einstein metric ω_{KE} of curvature -1 having mixed Poincaré and cone singularities along D.

Here, by ample, we mean that $c_1(K_X+D)$ contains a Kähler metric, or equivalently that K_X+D is a positive combination of ample $\mathbb{Q}$-divisors.

Proof. — We choose (h_i) and h_{K_X} some smooth hermitian metrics on the line bundles $\mathcal{O}_X(D_i)$ and $\mathcal{O}_X(K_X)$ respectively such that the product metric h on K_X+D has positive curvature ω_0, and up to renormalizing the metrics h_k, one can assume that $\omega := \omega_0 - \sum_{\{a_k=1\}} dd^c \log\log \frac{1}{|s_k|^2}$ defines a Kähler metric on X_{lc} with Poincaré singularities along D_{lc}; more precisely it is of Carlson-Griffiths type.
Lemma 2.1.6 shows that one can write

$$\omega^n = \frac{e^{-B}\Psi}{\prod |s_k|^2 \log^2 |s_k|^2}$$

with Ψ the smooth volume form on X attached to h_{K_X} (in particular $-\mathrm{Ric}\,\Psi = \Theta_{h_{K_X}}(K_X)$, the curvature of $(\mathcal{O}_X(K_X), h_{K_X})$), and $B \in \mathscr{C}^{k,\alpha}_{qc}(X \setminus D_{lc})$ for all k and α.
Now we use Theorem 2.3.1 with $f = B$, and ω as reference metric. We then get a Kähler

metric $\omega_{\mathrm{KE}} := \omega + dd^c\varphi$ on $X \setminus \mathrm{Supp}(D)$ with mixed Poincaré and cone singularities along D satisfying

$$(\omega + dd^c\varphi)^n = \frac{e^{\varphi+B}}{\prod_{j\in J}|s_j|^{2a_j}}\,\omega^n.$$

Therefore,

$$\begin{aligned}-\mathrm{Ric}\,(\omega_{\mathrm{KE}}) &= dd^c(\varphi+B) - dd^cB + \Theta_{h_{K_X}}(K_X) - \sum_{k\in K}\left(dd^c\log|s_k|^2 - dd^c\log\log\frac{1}{|s_k|^2}\right)\\ &\quad -\sum_{j\in J} dd^c\log|s_j|^{2a_j}\\ &= dd^c\varphi + \Theta(K_X) + \Theta(D_{lc}) + \Theta(D_{klt}) - \sum_{k\in K} dd^c\log\log\frac{1}{|s_k|^2}\\ &= \omega_{\mathrm{KE}}.\end{aligned}$$

Moreover, ω_{KE} has mixed Poincaré and cone singularities along D, so it is a Kähler-Einstein metric for the pair (X, D).
As for the uniqueness of ω_{KE}, it follows directly from Proposition 2.2.5. □

2.4. Proof of the main result

As we explained in the introduction, the natural strategy is to combine the approaches of [**CGP11**] and Kobayashi ([**Kob84**]). More precisely we will produce a sequence of Kähler metrics $(\omega_\varepsilon)_\varepsilon$ on $X \setminus D_{lc}$ having Poincaré singularities along D_{lc} and acquiring cone singularities along D_{klt} at the end of the process when $\varepsilon = 0$.

2.4.1. The approximation process. — We keep the notation of Theorem 2.3.1, so that ω is a Kähler form on X_{lc} of Carlson-Griffiths type; in particular it has Poincaré singularities along D_{lc}.
We define, for any sufficiently small $\varepsilon > 0$, a Kähler form ω_ε on X_{lc} by

$$\omega_\varepsilon := \omega + dd^c\psi_\varepsilon$$

where $\psi_\varepsilon = \frac{1}{N}\sum_{\{a_j<1\}}\chi_{j,\varepsilon}(\varepsilon^2 + |s_j|^2)$ for $\chi_{j,\varepsilon}$ functions defined by:

$$\chi_{j,\varepsilon}(\varepsilon^2 + t) = \frac{1}{\tau_j}\int_0^t \frac{(\varepsilon^2 + r)^{\tau_j} - \varepsilon^{2\tau_j}}{r}dr$$

for any $t \geqslant 0$. The important facts to remember about this construction are the following ones, extracted from [**CGP11**, section 3]:

- For N big enough, ω_ε dominates (as a current) a Kähler form on X because ω already does;
- ψ_ε is uniformly bounded (on X) in ε;
- When ε goes to 0, ω_ε converges on X_{lc} to ω_D having mixed Poincaré and cone singularities along D.

As ω_ε is a Kähler metric on X_{lc} with Poincaré singularities along D_{lc}, the case $J = \emptyset$ treated by Kobayashi ([**Kob84**]) and Tian-Yau ([**TY87**]), cf section 2.1.3, Theorem 2.1.7, enables us to find a smooth ω_ε-psh function φ_ε on X_{lc} satisfying:

$$(\omega_\varepsilon + dd^c\varphi_\varepsilon)^n = e^{\varphi_\varepsilon + F_\varepsilon}\omega_\varepsilon^n \tag{2.4.1}$$

where

$$F_\varepsilon = f + \psi_\varepsilon + \log\left(\frac{\omega^n}{\prod_{j\in J}(|s_j|^2+\varepsilon^2)^{a_j}\omega_\varepsilon^n}\right)$$

belongs to $\mathscr{C}^{k,\alpha}_{qc}(X_{lc})$ thanks to Lemma 2.1.6 and the assumptions on f. We may insist on the fact that the relation $F_\varepsilon \in \mathscr{C}^{k,\alpha}_{qc}(X_{lc})$ is only qualitative in the sense that we *a priori* don't have uniform estimates on $||F_\varepsilon||_{k,\alpha}$.
Besides, $\varphi_\varepsilon \in \mathscr{C}^{k,\alpha}_{qc}(X_{lc})$ (cf. [**Kob84**, section 3]) so that in particular, it is bounded on X_{lc}, $\omega_\varepsilon + dd^c\varphi_\varepsilon$ defines a complete Kähler metric on X_{lc}, and the Ricci curvature of $\omega_\varepsilon + dd^c\varphi_\varepsilon$ bounded (from below) if and only if the one of ω_ε is bounded (from below). Note that the bounds may *a priori* not be uniform in ε - however we will show that this is the case.
Once observed that ω_ε converges to a Kähler metric with mixed Poincaré and cone singularities along D, and that equation (2.4.1) is equivalent to

$$(\omega + dd^c(\varphi_\varepsilon + \psi_\varepsilon))^n = \frac{e^{f+(\varphi_\varepsilon+\psi_\varepsilon)}}{\prod_{j\in J}(|s_j|^2+\varepsilon^2)^{a_j}}\omega^n$$

the proof of our theorem boils down to showing that one can extract a subsequence of $(\varphi_\varepsilon)_\varepsilon$ converging to φ, smooth outside D, and such that $\omega + dd^c\varphi$ has the expected singularities along D.

2.4.2. Establishing estimates for φ_ε. — In view of the *a priori* estimates of section 2.1.4, we first need to find a bound $\sup|\varphi_\varepsilon| \leqslant C$. We will see at the beginning of section 2.4.2.3 that $\sup_\varepsilon \sup_X |F_\varepsilon|$ is finite. Therefore, using 2.1.9 with ω_ε as reference metric, we have the desired $\mathscr{C}^0$ estimate: $\sup|\varphi_\varepsilon| \leqslant \sup_\varepsilon \sup_X |F_\varepsilon|$. So it remains to check that (here uniformly means "uniformly in ε"):

(*i*) The bisectional curvature of $(X_{lc}, \omega_\varepsilon)$ is uniformly bounded from below;
(*ii*) F_ε is uniformly bounded;
(*iii*) The Laplacian of F_ε with respect to ω_ε, $D_{\omega_\varepsilon}F_\varepsilon$, is uniformly bounded.

Once we will have shown that conditions $(i)-(iii)$ hold, we will get the existence of $C>0$ such that for all $\varepsilon > 0, \mathrm{tr}_{\omega_\varepsilon}(\omega_\varepsilon + dd^c\varphi_\varepsilon) \leqslant C$ (by the remarks above, $\omega_\varepsilon + dd^c\varphi_\varepsilon$ is complete and will have Ricci curvature bounded from below so that the assumptions of Proposition 2.1.10 are fulfilled). Therefore, we will have $\omega_\varepsilon + dd^c\varphi_\varepsilon \leqslant C\omega_\varepsilon$. Furthermore, as φ_ε and F_ε will be bounded, the identity $(\omega_\varepsilon + dd^c\varphi_\varepsilon)^n = e^{\varphi_\varepsilon+F_\varepsilon}\omega_\varepsilon^n$ joint with the basic inequality $\det_{\omega_\varepsilon}(\omega_\varepsilon + dd^c\varphi_\varepsilon)\cdot \mathrm{tr}_{\omega_\varepsilon+dd^c\varphi_\varepsilon}(\omega_\varepsilon) \leqslant (\mathrm{tr}_{\omega_\varepsilon}(\omega_\varepsilon + dd^c\varphi_\varepsilon))^{n-1}$ (which amounts to saying that $\sum_{|I|=n-1}\prod_{i\in I}\lambda_i \leqslant (\sum_{i=1}^n \lambda_i)^{n-1}$) will imply that, up to increasing C, $\mathrm{tr}_{\omega_\varepsilon+dd^c\varphi_\varepsilon}(\omega_\varepsilon) \leqslant C$. Therefore,

$$C^{-1}\omega_\varepsilon \leqslant \omega_\varepsilon + dd^c\varphi_\varepsilon \leqslant C\omega_\varepsilon$$

and passing to the limit (after choosing a subsequence so that $(\varphi_\varepsilon)_\varepsilon$ converges to φ smooth outside $\mathrm{Supp}(D)$ - we skip some important details here, cf. section 2.4.3) our solution

$\omega_o + dd^c\varphi$ will have mixed Poincaré and cone singularities along D.

2.4.2.1. A precise expression of the metric. — Before we go any further, we have to give the explicit local expresssions of ω_ε. We recall that $D = \sum_{j\in J} a_j D_j + \sum_{k\in K} D_k$ for some disjoints sets $J, K \subset \mathbb{N}$, such that for all $j \in J$, $a_j < 1$. In the following, an index j (resp. k) will always be assumed to belong to J (resp. K).
First of all, pick some point $p_0 \in X$ sitting on $\mathrm{Supp}(D)$. We choose a neighborhood U of p_0 trivializing X and such that $\mathrm{Supp}(D) \cap U = \{\prod_{J_U} z_j \cdot \prod_{K_U} z_k = 0\}$ for some $J_U \subset J$ and $K_U \subset K$. Then if $i \notin J_U \cup K_U$, D_i does not meet U. To simplify the notations, one may suppose that $J_U = \{1, \dots, r\}$ and $K_U = \{r+1, \dots, d\}$. Finally, we stress the point that although $p_0 \in \mathrm{Supp}(D)$, all our computations will be done on $U \cap X_{lc} = U \setminus \mathrm{Supp}(D_{lc})$.

So as to simplify the computations, we will use the following (more or less basic) lemma, extracted from [**CGP11**, Lemma 4.1]:

Lemma 2.4.1. — *Let $(L_1, h_1), \dots, (L_d, h_d)$ be a set of hermitian line bundles on a compact Kähler manifold X, and for each index $j = 1, \dots, d$, let s_j be a section of L_j; we assume that the hypersurfaces*

$$Y_j := (s_j = 0)$$

are smooth, and that they have strictly normal intersections. Let $p_0 \in \bigcap Y_j$; then there exist a constant $C > 0$ and an open set $V \subset X$ centered at p_0, such that for any point $p \in V$ there exists a coordinate system $z = (z_1, \dots, z_n)$ at p and a trivialization θ_j for L_j such that:

(i) For $j = 1, \dots, d$, we have $Y_j \cap V = (z_j = 0)$;

(ii) With respect to the trivialization θ_j, the metric h_j has the weight φ_j, such that

$$\varphi_j(p) = 0, \quad d\varphi_j(p) = 0, \quad \left| \frac{\partial^{|\alpha|+|\beta|}\varphi_j}{\partial z_\alpha \partial \bar{z}_\beta}(p) \right| \leqslant C_{\alpha,\beta}$$

for all multi indexes α, β.

Up to shrinking the neighborhood V, we may assume that each coordinate system $(z_1, \dots, z_n)$ for V, as given in Lemma 2.4.1, satisfies $\sum_i |z_i|^2 \leqslant 1/2$. Moreover, in order to make the notations clearer, we define, for $i \in \{1, \dots, n\}$, a non-negative function on V (depending on ε) by

$$A(i) = \begin{cases} (|z_i|^2 + \varepsilon^2)^{a_i/2} & \text{if } i \in \{1, \dots, r\}; \\ |z_i| \log \frac{1}{|z_i|^2} & \text{if } i \in \{r+1, \dots, d\}; \\ 1 & \text{if } i < d. \end{cases}$$

Now, for $i, j, k, l \in \{1, \dots, n\}$, we simply set $A(i, j, k, l) := A(i)A(j)A(k)A(l)$.

We first want to check that the holomorphic bisectional curvature of ω_ε is bounded from below, that is

$$\Theta_{\omega_\varepsilon}(T_X) \geqslant -C\omega_\varepsilon \otimes \mathrm{Id}_{T_X} \tag{2.4.2}$$

for some $C > 0$ independent of ε, and where $\Theta_{\omega_\varepsilon}(T_X)$ denotes the curvature tensor of the holomorphic tangent bundle of $(X_{lc}, \omega_\varepsilon)$. It is useful for the following to reformulate the (intrinsic) condition (2.4.2) in terms of local coordinates. Namely, the inequality in (2.4.2) amounts to saying that the following inequality holds:

$$\sum_{p,q,r,s} R_{p\bar{q}r\bar{s}}(z) v_p \overline{v_q} w_r \overline{w_s} \geqslant -C|v|^2_{\omega_\varepsilon}|w|^2_{\omega_\varepsilon} \tag{2.4.3}$$

for any vector fields $v = \sum_p v_p \frac{\partial}{\partial z_p}$ and $w = \sum_r w_r \frac{\partial}{\partial z_r}$.

The notation in the above relations is as follows: in local coordinates, we write

$$\omega_\varepsilon = \frac{i}{2}\sum_{p,q} g_{p\bar{q}}\, dz_p \wedge d\bar{z}_q;$$

(so that the $g_{p\bar{q}}$'s actually depend on ε, but we choose not to let it appear in the notations so as to make them a bit lighter) and the corresponding components of the curvature tensor are

$$R_{p\bar{q}r\bar{s}} := -\frac{\partial^2 g_{p\bar{q}}}{\partial z_r \partial \bar{z}_s} + \sum_{k,l} g^{k\bar{l}} \frac{\partial g_{p\bar{k}}}{\partial z_r}\frac{\partial g_{l\bar{q}}}{\partial \bar{z}_s}.$$

Looking at the local expression of ω_ε makes it clear that there exists $C>0$ independent of ε such that on V, $C^{-1}\omega_{D,\varepsilon} \leqslant \omega_\varepsilon \leqslant C\,\omega_{D,\varepsilon}$, where

$$\omega_{D,\varepsilon} := \sum_{j=1}^{r} \frac{idz_j \wedge d\bar{z}_j}{(|z_j|^2+\varepsilon^2)^{a_j}} + \sum_{k=r+1}^{d} \frac{idz_k \wedge d\bar{z}_k}{|z_k|^2 \log^2|z_k|^2} + \sum_{l=d}^{n} idz_l \wedge d\bar{z}_l$$

Therefore, if $v = \sum_p v_p \frac{\partial}{\partial z_p}$ satisfies $|v|_{\omega_\varepsilon} = 1$, then for each p, $|v_p| \leqslant A(p)$. We are now going to show the following two facts, which will ensure that the holomorphic bisectional curvature of ω_ε is bounded from below:

- (i) For every four-tuple (p,q,r,s) with $\#\{p,q,r,s\} \geqslant 2$, we have $A(p,q,r,s)|R_{p\bar{q}r\bar{s}}(z)| \leqslant C$;
- (ii) For every p, and every ω_ε-unitary vector fields v, w, $|v_p|^2_{\omega_\varepsilon}|w_p|^2_{\omega_\varepsilon} R_{p\bar{p}p\bar{p}} \geqslant -C$.

In order to prove $(i)-(ii)$, we have to give a precise expression of the metric ω_ε in some coordinate chart. We will use the coordinates given by Lemma 2.4.1, which will simplify the computations a lot. We remind that $\omega_\varepsilon = \omega + dd^c\psi_\varepsilon$, and according to [**CGP11**, equation (21)] and Definition 2.1.4 (or [**Gri76**, pp. 50-51]), the components $g_{p\bar{q}}$ of ω_ε are given by:

$$\begin{aligned} g_{p\bar{q}} \;=\;& u_{p\bar{q}} + \frac{\delta_{pq,J}e^{-\varphi_p}}{(|z_p|^2e^{-\varphi_p}+\varepsilon^2)^{a_p}} + \delta_{p,J}e^{-\varphi_p}\frac{\bar{z}_p\overline{\alpha_{qp}}}{(|z_p|^2e^{-\varphi_p}+\varepsilon^2)^{a_p}} + \delta_{q,J}e^{-\varphi_q}\frac{z_q\alpha_{qp}}{(|z_q|^2e^{-\varphi_q}+\varepsilon^2)^{a_q}} \\ &+ \sum_{j\in J}\frac{|z_j|^2\beta_{jpq}}{(|z_j|^2e^{-\varphi_j}+\varepsilon^2)^{a_j}}\Big((|z_j|^2e^{-\varphi_j}+\varepsilon^2)^{1-a_j}-\varepsilon^{2(1-a_j)}\Big)\frac{\partial^2\varphi_j}{\partial z_p\partial\bar{z}_q} \\ &+ \delta_{pq,K}\frac{idz_p\wedge d\bar{z}_p}{|z_p|^2\log^2|z_p|^2} + \frac{\delta_{p,K}\lambda_p}{z_p\log^2|z_p|^2} + \frac{\delta_{q,K}\mu_q}{\bar{z}_q\log^2|z_q|^2} + \sum_{k=r+1}^{d}\frac{\nu_k}{\log|z_k|^2} \end{aligned} \tag{2.4.4}$$

where $u_{pq}, \alpha_{pq}, \beta_{jpq}, \lambda_p, \mu_q, \nu_k$ are smooth functions on X (more precisely on the whole neighborhood V of p in X given by Lemma 2.4.1). Moreover, α, λ, μ (resp. β) are functions of the partial derivatives of the φ_i's; in particular, they vanish at the given point p at order at least 1 (resp. 2). Finally, we use the notation $\delta_{p,J} = \delta_{p\in J}$ and $\delta_{pq,J} = \delta_{pq}\delta_{p\in J}$ (*idem* for K instead of J).

2.4.2.2. Bounding the curvature from below. — First of all, using (2.4.4), and remembering that $\alpha, \beta, \lambda, \mu$, vanish at p, on can give a precise 0-order estimate on the metric (more precisely on the inverse matrix of the metric), which is a straightforward generalization of [**CGP11**, Lemma 4.2]:

Lemma 2.4.2. — *In our setting, and for $|z|^2 + \varepsilon^2$ sufficiently small, we have at the previously chosen point p:*

(i) For all $i \in \{1, \dots, n\}$, $g^{i\bar{i}} = A(i)^2(1 + \mathcal{O}(A(i)^2))$;
(ii) For all $j, k \in \{1, \dots, n\}$ such that $j \neq k$, $g^{j\bar{k}} = \mathcal{O}(A(j,k)^2)$.

We insist on the fact that the $\mathcal{O}$ symbol refers to the expression $|z|^2 + \varepsilon^2 = |z_1|^2 + \dots + |z_n|^2 + \varepsilon^2$ going to zero.

To bound the curvature, we will essentially have to deal with the Poincaré part of ω_ε, the other cone part being almost already treated in [**CGP11**]. We could use the fact that (X_{lc}, ω) has bounded geometry at any order (cf section 2.1.3), but as mixed terms involving the (regularized) cone metric will appear – which is not known to be of bounded geometry–, we prefer to give the explicit computations for more clarity.
For λ and μ any smooth functions on V, there exist smooth functions $\lambda_1, \lambda_2, \dots$ and $\mu_1, \mu_2, \dots$

such that for any $k \in K$:

$$\begin{aligned}
\frac{\partial}{\partial z_k}\left(\frac{\lambda}{z_k \log^2|z_k|^2}\right) &= \frac{\lambda_1}{z_k\log^2|z_k|^2} + \frac{\lambda_2}{z_k^2\log^2|z_k|^2} + \frac{\lambda_3}{z_k^2\log^3|z_k|^2} = \mathcal{O}\left(\frac{1}{|z_k|^2\log^2|z_k|^2}\right)\\
\frac{\partial}{\partial \bar z_k}\left(\frac{\lambda}{z_k \log^2|z_k|^2}\right) &= \frac{\lambda_4}{z_k\log^2|z_k|^2} + \frac{\lambda_5}{|z_k|^2\log^3|z_k|^2} = \mathcal{O}\left(\frac{1}{|z_k|^2\log^3|z_k|^2}\right)\\
\frac{\partial^2}{\partial z_k\partial \bar z_k}\left(\frac{\lambda}{z_k \log^2|z_k|^2}\right) &= \frac{\lambda_6}{z_k\log^2|z_k|^2} + \frac{\lambda_7}{|z_k|^2\log^3|z_k|^2} + \frac{\lambda_8}{z_k^2\log^2|z_k|^2} +\\
&+ \frac{\lambda_9}{z_k|z_k|^2\log^3|z_k|^2} + \frac{\lambda_{10}}{z_k^2\log^3|z_k|^2} + \frac{\lambda_{11}}{z_k|z_k|^2\log^4|z_k|^2}\\
&= \mathcal{O}\left(\frac{1}{|z_k|^3\log^3|z_k|^2}\right)\\
\frac{\partial}{\partial z_k}\left(\frac{\mu}{\log|z_k|^2}\right) &= \frac{\mu_1}{\log|z_k|^2} + \frac{\mu_2}{z_k\log^2|z_k|^2} = \mathcal{O}\left(\frac{1}{|z_k|\log^2|z_k|^2}\right)\\
\frac{\partial}{\partial \bar z_k}\left(\frac{\mu}{\log|z_k|^2}\right) &= \frac{\mu_3}{\log|z_k|^2} + \frac{\mu_4}{\bar z_k\log^2|z_k|^2} = \mathcal{O}\left(\frac{1}{|z_k|\log^2|z_k|^2}\right)\\
\frac{\partial^2}{\partial z_k\partial \bar z_k}\left(\frac{\mu}{\log|z_k|^2}\right) &= \frac{\mu_5}{\log|z_k|^2} + \frac{\mu_6}{\bar z_k\log^2|z_k|^2} + \frac{\mu_7}{z_k\log^2|z_k|^2} + \frac{\mu_8}{|z_k|^2\log^3|z_k|^2}\\
&= \mathcal{O}\left(\frac{1}{|z_k|^2\log^3|z_k|^2}\right)\\
\frac{\partial}{\partial z_k}\left(\frac{1}{|z_k|^2\log^2|z_k|^2}\right) &= \frac{-1}{z_k|z_k|^2\log^2|z_k|^2} + \frac{-2}{z_k|z_k|^2\log^3|z_k|^2} = \mathcal{O}\left(\frac{1}{|z_k|^3\log^2|z_k|^2}\right)\\
\frac{\partial^2}{\partial z_k\partial \bar z_k}\left(\frac{1}{|z_k|^2\log^2|z_k|^2}\right) &= \frac{1}{|z_k|^4\log^2|z_k|^2} + \frac{4}{|z_k|^4\log^3|z_k|^2} + \frac{6}{|z_k|^4\log^4|z_k|^2}
\end{aligned}$$

As we are mostly interested in the Poincaré part of the metric g, we will write $g = g^{(P)} + g^{(C)}$ its decomposition into the Poincaré and the cone part (cf. the expression (2.4.4)). Moreover, we write $g^{(P)} = \gamma^0 + \gamma$ where $\gamma^0 = \sum_{k\in K}\frac{idz_k\wedge d\bar z_k}{|z_k|^2\log^2|z_k|^2}$. Therefore, if $k \neq l$, $g^{(P)}_{k\bar l} = \gamma_{k\bar l}$, and the computations above lead to (for every $k, l, r, s \in K$):

$$\frac{\partial g^{(P)}_{k\bar l}}{\partial z_k} = \mathcal{O}\left(\frac{1}{A(k)^2A(l)}\right) \quad \text{if } k \neq l \tag{2.4.5}$$

$$\frac{\partial^2 g^{(P)}_{k\bar l}}{\partial z_k\partial \bar z_r} = \mathcal{O}\left(\frac{1}{A(k)^2A(r,l)}\right) \quad \text{if } k \neq l \tag{2.4.6}$$

$$\frac{\partial \gamma_{k\bar l}}{\partial z_r} = \mathcal{O}\left(\frac{1}{A(k,l,r)}\right) \tag{2.4.7}$$

$$\frac{\partial^2 \gamma_{k\bar l}}{\partial z_r\partial \bar z_s} = \mathcal{O}\left(\frac{1}{A(k,l,r,s)}\right) \tag{2.4.8}$$

Furthermore, we may note that if $\{p, q, r, s\} \cap J = \emptyset$, then we can see from the expression (2.4.4) that $\frac{\partial g_{p\bar q}}{\partial z_r} = \frac{\partial g^{(P)}_{p\bar q}}{\partial z_r} + \mathcal{O}(1)$ as well as $\frac{\partial^2 g_{p\bar q}}{\partial z_r\partial \bar z_s} = \frac{\partial g^{(P)}_{p\bar q}}{\partial z_r\partial \bar z_s} + \mathcal{O}(1)$. From this, (2.4.5)-(2.4.6) and Lemma 2.4.2, we deduce that for every $p, q, r, s \in K$ such that $p \neq q$, the expression

$A(p,q,r,s)R_{p\bar{q}r\bar{s}}(z)$ is uniformly bounded in $z \in V \cap X_{lc}$.

So it remains to study the terms of the form $R_{p\bar{p}r\bar{s}}$ for $p, r, s \in K$. And as mentionned in the last paragraph, the terms in the curvature tensor coming from the cone part (or the smooth part) do not play any role here, so we have:

$$\begin{aligned} R_{p\bar{p}r\bar{s}} &= -\frac{\partial^2 g_{p\bar{p}}}{\partial z_r \partial \bar{z}_s} + \sum_{1 \leqslant k,l \leqslant n} g^{k\bar{l}} \frac{\partial g_{p\bar{l}}}{\partial z_r} \frac{\partial g_{k\bar{p}}}{\partial \bar{z}_s} \\ &= -\frac{\partial^2}{\partial z_r \partial \bar{z}_s} \left(\frac{1}{|z_p|^2 \log^2 |z_p|^2} \right) - \frac{\partial^2 \gamma_{p\bar{p}}}{\partial z_r \partial \bar{z}_s} + \sum_{1 \leqslant k,l \leqslant n} g^{k\bar{l}} \frac{\partial g^{(P)}_{p\bar{l}}}{\partial z_r} \frac{\partial g^{(P)}_{k\bar{p}}}{\partial \bar{z}_s} + \mathcal{O}(1) \end{aligned}$$

Using (2.4.5)-(2.4.8) and Lemma 2.4.2, we see that the only possibly unbounded terms (when multiplied by $A(p)^2 A(r,s)$) appearing in the expansion of $R_{p\bar{p}r\bar{s}}$ are coming from γ_0. More precisely, these are the following ones, appearing in $R_{p\bar{p}p\bar{p}}$ only:

$$(2.4.9) \qquad -\frac{\partial^2}{\partial z_p \partial \bar{z}_p} \left(\frac{1}{|z_p|^2 \log^2 |z_p|^2} \right) + \sum_{p \in \{k,l\}} g^{k\bar{l}} \frac{\partial g^{(P)}_{p\bar{l}}}{\partial z_p} \frac{\partial g^{(P)}_{k\bar{p}}}{\partial \bar{z}_p}$$

Let us now expand the terms under the sum:

$$(2.4.10) \qquad \frac{\partial g^{(P)}_{k\bar{p}}}{\partial z_p} = \mathcal{O}\left(\frac{1}{|z_p|^2 \log^3 |z_p|^2} \right) \qquad \text{if } k \neq p$$

$$(2.4.11) \qquad \frac{\partial g^{(P)}_{p\bar{p}}}{\partial z_p} = \frac{-1}{z_p |z_p|^2 \log^2 |z_p|^2} + \frac{-2}{z_p |z_p|^2 \log^3 |z_p|^2} + \mathcal{O}\left(\frac{1}{|z_k|^2 \log^3 |z_k|^2} \right)$$

$$(2.4.12) \qquad \left| \frac{\partial g^{(P)}_{p\bar{p}}}{\partial z_p} \right|^2 = \frac{1}{|z_p|^6 \log^4 |z_p|^2} \left(1 + \frac{4}{\log |z_k|^2} + \frac{4}{\log^2 |z_k|^2} + \mathcal{O}(|z_k|) \right)$$

Now, if we combine Lemma 2.4.2 with (2.4.10)-(2.4.11), we see that the remaining possibly unbounded terms (after multiplying by $A(p)^4$) appearing in (2.4.9) are

$$-\frac{\partial^2}{\partial z_p \partial \bar{z}_p} \left(\frac{1}{|z_p|^2 \log^2 |z_p|^2} \right) + g^{p\bar{p}} \frac{\partial g^{(P)}_{p\bar{p}}}{\partial z_p} \frac{\partial g^{(P)}_{p\bar{p}}}{\partial \bar{z}_p}$$

which, thanks to point (i) of Lemma 2.4.2 and (2.4.12), happens to be a $\mathcal{O}\left(\frac{1}{|z_p|^4 \log^4 |z_p|^2} \right)$, which finishes to prove that for every $p, q, r, s \in K$, the expression $A(p,q,r,s)R_{p\bar{q}r\bar{s}}(z)$ is uniformly bounded in $z \in V \cap X_{lc}$.

Now we may look at the terms $R_{p\bar{q}r\bar{s}}$ where $p, q \in K$ but $r, s \notin K$. If $r, s \notin J$, then $A(p,q,r,s)R_{p\bar{q}r\bar{s}}(z) = A(p,q)R_{p\bar{q}r\bar{s}}(z)$ is uniformly bounded in $z \in V \cap X_{lc}$ as we can see by looking at the expression of the metric (2.4.4). So now we may suppose that r or s belongs to J. The only term in the metric which may cause trouble is $\sum_{j \in J} \frac{|z_j|^2 \beta_{jpq}}{(|z_j|^2 e^{-\varphi_j} + \varepsilon^2)^{a_j}} + \left((|z_j|^2 e^{-\varphi_j} + \varepsilon^2)^{1-a_j} - \varepsilon^{2(1-a_j)} \right) \frac{\partial^2 \varphi_j}{\partial z_p \partial \bar{z}_q}$. But Lemma 2.4.2 enables us to use the computations of [**CGP11**, section 4.3] word for word, so as to show that

$A(p,q,r,s)R_{p\bar{q}r\bar{s}}(z)$ is uniformly bounded in $z \in V \cap X_{lc}$.

The next step in bounding the curvature of ω_ε from below consists now in looking at the terms $R_{p\bar{q}r\bar{s}}$ for $p,q \in J$. Then the terms in $g_{p\bar{q}}$ coming from the Poincaré part are of the form $\sum_k \frac{\nu_k}{\log|z_k|^2}$ as (2.4.4) shows. These terms are uniformly bounded in $V \cap X_{lc}$, as well as their derivatives with respect to the variables $z_r, \bar{z}_s$ as long as $r, s \notin K$; in that that case [**CGP11**, sections 4.3-4.4] gives us the expected lower bound for $A(p,q,r,s)R_{p\bar{q}r\bar{s}}$. If now $r \in K$, then we saw earlier that $A(r)\frac{\partial}{\partial z_r}\left(\frac{\nu_r}{\log|z_r|^2}\right)$, $A(s)\frac{\partial}{\partial \bar{z}_s}\left(\frac{\nu_s}{\log|z_k|^2}\right)$, $A(r)^2\frac{\partial^2}{\partial z_r \partial \bar{z}_r}\left(\frac{\nu_r}{\log|z_r|^2}\right)$ are bounded functions in $V \cap X_{lc}$, so that, using Lemma 2.4.2, the boundedness of $A(p,q,r,s)R_{p\bar{q}r\bar{s}}$ is equivalent to the one of $A(p,q,r,s)R^{g^{(C)}}_{p\bar{q}r\bar{s}}$ whenever $p,q \in J$. And by [**CGP11**, section 4.3], we know the existence of this bound (which is an upper and lower bound, as $\#\{p,q,r,s\} \geqslant 2$) .

Finally, for the last step, we need to look at mixed terms $R_{p\bar{q}r\bar{s}}$ for $p \in K$ and $q \in J$ (or one of those not belonging to $J \cup K$). As $p \neq q$, the operators $A(r)\frac{\partial}{\partial z_r}, A(s)\frac{\partial}{\partial \bar{z}_s}$ and $A(r,s)\frac{\partial^2}{\partial z_r \partial \bar{z}_r}$ map $g_{p\bar{q}}$ to a bounded function, as can be checked separately for $g^{(P)}$ (cf. the previous computations) and $g^{(C)}$ (cf. [**CGP11**, section 4.3]).
So we are done: ω_ε has holomorphic bisectional curvature *uniformly* bounded from below on X_{lc}.

2.4.2.3. Bounding the ω_ε-Laplacian of F_ε. — Remember that

$$F_\varepsilon = f + \psi_\varepsilon + \log\left(\frac{\omega^n}{\prod_{j\in J}(|s_j|^2+\varepsilon^2)^{a_j}\omega_\varepsilon^n}\right)$$

At the point x (which is point p of Lemma 2.4.1) , the $(p,\bar{q})$ component of $\omega_\varepsilon(x)$ is

$$\begin{aligned} g_{p\bar{q}}(x) &= u_{p\bar{q}}(x) + \frac{\delta_{pq,J}}{(|z_p|^2+\varepsilon^2)^{a_p}} + \sum_{j\in J}\left((|z_j|^2+\varepsilon^2)^{1-a_j} - \varepsilon^{2(1-a_j)}\right)\frac{\partial^2\varphi_j}{\partial z_p \partial \bar{z}_q}(x) + \\ &+ \delta_{pq,K}\frac{idz_p \wedge d\bar{z}_p}{|z_p|^2\log^2|z_p|^2} + \sum_{k=r+1}^{d}\frac{\nu_k}{\log|z_k|^2} \end{aligned}$$

whereas the $(p,\bar{q})$ component of $\omega(x)$ is

$$g^{(P)}_{p\bar{q}}(x) = u_{p\bar{q}}(x) + \delta_{pq,K}\frac{idz_p \wedge d\bar{z}_p}{|z_p|^2\log^2|z_p|^2} + \sum_{k=r+1}^{d}\frac{\nu_k}{\log|z_k|^2}$$

Expanding the determinant of those metrics makes it clear that there exists $C>0$ such that

$$C^{-1} \leqslant \frac{\omega^n}{\prod_{j\in J}(|s_j|^2+\varepsilon^2)^{a_j}\omega_\varepsilon^n} \leqslant C$$

so that F_ε is bounded on X_{lc}.

Let us now get to bounding $\Delta_{\omega_\varepsilon}F_\varepsilon$. Actually we will show that $\pm dd^c F_\varepsilon \leqslant C\omega_\varepsilon$ for some uniform $C>0$, which is stronger than just bounding the ω_ε-Laplacian of F_ε, but we need this strengthened bound if we want to produce Kähler-Einstein metrics by resolving our

Monge-Ampère equation. There are three terms in F_ε, namely f, ψ_ε and $\log F_\varepsilon$ where

$$F_\varepsilon = \frac{\omega^n}{\prod_{j\in J}(|s_j|^2+\varepsilon^2)^{a_j}\omega_\varepsilon^n}$$

The first two terms are easy to deal with: indeed, there exists $C>0$ (independent of ε) such that $\omega_\varepsilon \geqslant C^{-1}\omega$ on X_{lc}. Therefore, if one chooses M such that $M\omega \pm dd^c f > 0$ (the assumptions on f give the existence of such an M), then $dd^c f \leqslant CM\omega_\varepsilon$. Moreover, $\omega_\varepsilon = \omega + dd^c\psi_\varepsilon > 0$ so that $\pm dd^c\psi_\varepsilon \leqslant \max(C,1)\omega_\varepsilon$. Therefore it only remains to bound $dd^c \log F_\varepsilon$ now.

We will use the following basic identities, holding for any smooth functions $f>0$ and u, v on some open subset of $U \subset X$:

$$dd^c \log f = \frac{1}{f}dd^c f + \frac{1}{f^2} df\wedge d^c f \tag{2.4.13}$$

$$dd^c\left(\frac{1}{f}\right) = -\frac{1}{f^2}dd^c f + \frac{2}{f^3}df\wedge d^c f \tag{2.4.14}$$

$$dd^c(uv) = u\, dd^c v + v\, dd^c u + du\wedge d^c v - d^c u\wedge dv \tag{2.4.15}$$

$$\nabla(uv) = (\nabla u)\, v + u\,(\nabla v) \tag{2.4.16}$$

We just saw that F_ε is bounded below by some fixed constant $C^{-1}>0$ on X_{lc}, so that by (2.4.13), $\pm dd^c \log F_\varepsilon$ will be dominated by some fixed multiple of ω_ε if we show that both $\pm dd^c F_\varepsilon \leqslant C\omega_\varepsilon$ and $|\nabla_\varepsilon F_\varepsilon|_\omega \leqslant C$ for some uniform $C>0$ (the last term denotes the norm computed with respect to ω of the ω_ε-gradient of F_ε, defined as usual by $dF_\varepsilon(X) = \omega_\varepsilon(\nabla_\varepsilon F_\varepsilon, X)$ for every vector field X). For convenience, we will split the computation by writing

$$F_\varepsilon = \left(\prod_{j\in J}(|s_j|^2+\varepsilon^2)^{a_j}\cdot \prod_{k\in K}|s_k|^2\log^2|s_k|^2\cdot\omega_\varepsilon^n\right)^{-1}\cdot\left(\prod_{k\in K}|s_k|^2\log^2|s_k|^2\cdot\omega^n\right) \tag{2.4.17}$$

By (2.4.14)-(2.4.15), we only need to check that the gradient ∇_ε of the terms inside the parenthesis is bounded, and that their $\pm dd^c$ is dominated by some fixed multiple of ω_ε. Let us begin with the second one, which is simpler:

Lemma 2.4.3. — *Let ω be a Kähler form of Carlson-Griffiths type on X_{lc}, and let ω_0 be some smooth Kähler form on X. We set*

$$V = \left(\prod_{k\in K}|s_k|^2\log^2|s_k|^2\right)\cdot\frac{\omega^n}{\omega_0^n}$$

Then there exists $C>0$ such that $\pm V$ is $C\omega$-psh on X_{lc}.

Proof. — We write, with our usual coordinates (cf Lemma 2.4.1) :

$$(2.4.18)\omega^n = \prod_{k\in K}\frac{1}{|z_k|^2\log^2|z_k|^2}\left(1+\sum_{K_i\subset K}A_i\prod_{k_i\in K_i}\frac{1}{\log|z_{k_i}|^2}\right) + \sum_{K_j,K_l,K_m,K_p\subset K}A_{jlmp}\prod_{k_j\in K_j}\frac{1}{|z_{k_j}|^2\log^2|z_{k_j}|^2}\cdot\prod_{k_l\in K_l}\frac{1}{z_{k_l}\log^2|z_{k_l}|^2}\cdot \cdot\prod_{k_m\in K_m}\frac{1}{\bar z_{k_m}\log^2|z_{k_m}|^2}\cdot\prod_{k_p\in K_p}\frac{1}{\log|z_{k_p}|^2}\cdot\Omega$$

for Ω some smooth volume form on X and where the second sum is taken over the subsets K_j, K_l, K_m, K_p of K that are disjoint, and where A_i, A_{jlmp} are smooth functions on the whole X. Let us apply the operators $A(i,j)\frac{\partial}{\partial z_i\partial\bar z_j}$ and $g^{i\bar j}\frac{\partial}{\partial z_i}\cdot\frac{\partial}{\partial\bar z_j}$ to $\frac{1}{\log|z_k|^2}, z_k, \bar z_k, |z_k|^2\log|z_k|^2$ and $|z_k|^2\log^2|z_k|^2$, and check that we obtain bounded functions. We already did it for the first term, so we only have to compute:

$$\begin{aligned}
\frac{\partial}{\partial z_k}(|z_k|^2\log|z_k|^2) &= \bar z_k\log|z_k|^2+\bar z_k=\mathcal{O}(1)\\
\frac{\partial^2}{\partial z_k\partial\bar z_k}(|z_k|^2\log|z_k|^2) &= \log|z_k|^2+2=\mathcal{O}\left(\frac{1}{|z_k|^2\log^2|z_k|^2}\right)\\
\frac{\partial}{\partial z_k}(|z_k|^2\log^2|z_k|^2) &= \bar z_k\log^2|z_k|^2+2\bar z_k\log|z_k|^2=\mathcal{O}(1)\\
\frac{\partial^2}{\partial z_k\partial\bar z_k}(|z_k|^2\log^2|z_k|^2) &= \log^2|z_k|^2+4\log|z_k|^2+2=\mathcal{O}\left(\frac{1}{|z_k|^2\log^2|z_k|^2}\right)
\end{aligned}$$

This shows that the ω_ε-gradient of these factors (denote them generically κ) is bounded. As for $dd^c\kappa$, the previous computations show that in coordinates, its (i,j)-th term is uniformly bounded by $CA(i,j)$ for every i,j (this is actually stronger than saying that it becomes bounded when multiplied with $g^{i\bar j}$, condition which would however be sufficient to show that the ω_ε-Laplacian is bounded). Therefore, as the matrix of ω_ε can be written $\mathrm{diag}(A(1)^2,\dots,A(n)^2)+\mathcal{O}(1)$ in coordinates, and using the Cauchy-Schwarz inequality, one easily obtains $C>0$ such that $\pm dd^c\kappa\leqslant C\omega_\varepsilon$.
In fact, once we we saw that the only singular terms were $\frac{1}{\log|z_k|^2}, |z_k|^2\log|z_k|^2$ and $|z_k|^2\log^2|z_k|^2$, we could have used the usual quasi-coordinates as in 2.1.6 to conclude. □

Let us now get to the term inside the first parenthesis of (2.4.17). For this, notice that in the expansion of ω_ε^n, we find the terms of (2.4.18) multiplied by terms of the form

$$C(z)+\sum_{I\subsetneq J}A_I(z)\prod_{i\in I}(|z_i|^2e^{-\varphi_i}+\varepsilon^2)^{a_i}$$

where $C(z)$ and $A_I(z)$ are sums of terms of the form

$$B(z)\prod_{j_l\in J_l}[(|z_{j_l}|^2e^{-\varphi_{j_l}}+\varepsilon^2)^{1-a_{j_l}}-\varepsilon^{2(1-a_{j_l})}]\cdot\prod_{j\in J_k}\frac{z_{j_k}\alpha_{j_k}}{(|z_{j_k}|^2e^{-\varphi_{j_k}}+\varepsilon^2)^{\lambda_{j_k}a_{j_k}}}\cdot\;\cdots$$

$$\cdots\quad\cdot\prod_{j\in J_m}\frac{\bar z_{j_m}\bar\alpha_{j_m}}{(|z_{j_m}|^2e^{-\varphi_{j_m}}+\varepsilon^2)^{\lambda_{j_m}a_{j_m}}}\prod_{j_p\in J_p}\frac{|z_{j_p}|^2\beta_{j_p}}{(|z_{j_p}|^2e^{-\varphi_{j_p}}+\varepsilon^2)^{a_{j_p}}}$$

where I, J_l, J_k, J_m, J_p are disjoint subsets of J, and where $B(z)$ is smooth independent of ε, α_j is smooth and vanishes at x, β_j is smooth and vanishes at order at least 2 at p, and $\lambda_j \in \{0, 1/2\}$. And now, using Lemma 2.4.2 and [**CGP11**, section 4.5] (we must slightly change the argument therein as said above to control the dd^c with respect to ω_ε and not only $\Delta_{\omega_\varepsilon}$), we can conclude that the appropriate dd^c (resp. gradients) of those quantities are dominated by $C\omega_\varepsilon$ (resp. bounded). Combining this with the previous computations, we deduce that $\Delta_{\omega_\varepsilon} F_\varepsilon$ is bounded on the whole X_{lc}.

2.4.3. End of the proof. — Remember that we wish to extract from the sequence of smooth metrics $\omega_\varepsilon + dd^c\varphi_\varepsilon$ on X_{lc} some subsequence converging to a smooth metric on $X \setminus \mathrm{Supp}(D)$. In order to do this, we need to have *a priori* $\mathscr{C}^k$ estimates for all k. The usual bootstrapping argument for the Monge-Ampère equation allows us to deduce those estimates from the $\mathscr{C}^{2,\alpha}$ ones for some $\alpha \in]0,1[$. The crucial fact here is that we have at our disposal the following *local* result, taken from [**GT77**] (see also [**Siu87**], [**Bło11**, Theorem 5.1]), which gives interior estimates. It is a consequence of Evans-Krylov's theory:

Theorem 2.4.4. — *Let u be a smooth psh function in an open set $\Omega \subset \mathbb{C}^n$ such that $f := \det(u_{i\bar{j}}) > 0$. Then for any $\Omega' \Subset \Omega$, there exists $\alpha \in]0,1[$ depending only on n and on upper bounds for $||u||_{\mathscr{C}^0(\Omega)}$, $\sup_\Omega \Delta\varphi, ||f||_{\mathscr{C}^{0,1}(\Omega)}, 1/\inf_\Omega f$, and $C > 0$ depending in addition on a lower bound for $d(\Omega', \partial\Omega)$ such that:*

$$||u||_{\mathscr{C}^{2,\alpha}(\Omega')} \leqslant C.$$

In our case, we choose some point p outside the support of the divisor D, and consider two coordinate open sets $\Omega' \subset \Omega$ containing p, but not intersecting $\mathrm{Supp}(D)$. In that case, we may find a smooth Kähler metric ω_p on Ω such that on Ω', the covariant derivatives at any order of ω_ε are uniformly bounded (in ε) with respect to ω_p. Then one may take $u = \varphi_\varepsilon$ in the previous theorem, and one can easily check that there are common upper bounds (i.e. independent of ε) for all the quantities involved in the statement. This finishes to show the existence of uniform *a priori* $\mathscr{C}^{2,\alpha}(\Omega')$ estimates for φ_ε.

As we mentioned earlier, the ellipticity of the Monge-Ampère operator automatically gives us local *a priori* $\mathscr{C}^k$ estimates for φ_ε, which ends to provide a smooth function φ on $X \setminus \mathrm{Supp}(D)$ (extracted from the sequence $(\varphi_\varepsilon)_\varepsilon$) such that $\omega_\infty = \omega + dd^c\varphi$ defines a smooth metric outside $\mathrm{Supp}(D)$ satisfying

$$(\omega + dd^c\varphi)^n = \frac{e^{\varphi+f}}{\prod_{j\in J} |s_j|^{2a_j}}\, \omega^n.$$

Moreover, the strategy explained at the beginning of the previous section 2.4.2 and set up all along the section shows that this metric φ has mixed Poincaré and cone singularities along D, so this finishes the proof of the main theorem.

2.4.4. Remarks. — It could also be interesting to study the following equation:

$$(\omega + dd^c\varphi)^n = \frac{e^{f}}{\prod_{j\in J} |s_j|^{2a_j}}\, \omega^n$$

where ω is of Carlson-Griffith's type, and asked whether its eventual solutions have mixed Poincaré and cone singularities. This equation has been recently studied and solved by H.

Auvray in [**Auv11**, Theorem 4] in the case where $D_{klt} = 0$ (the "logarithmic case"), and for f vanishing at some order along D. To adapt his results, one would need to show that one can make a choice of ψ_ε so that F_ε vanishes along D_{lc} at some fixed order, what we have been unable to do so far.

2.5. A vanishing theorem for holomorphic tensor fields

Given a pair (X, D), where X is a compact Kähler manifold and $D = \sum a_i D_i$ a $\mathbb{R}$-divisor with simple normal crossing support such that $0 \leqslant a_i \leqslant 1$, there are many natural ways to construct holomorphic tensors attached to (X, D).
To begin with, one defines the tensor fields on a manifold M, which are contravariant of degree r and covariant of degree s as follows

$$T^r_s M := (\otimes^r T_M) \otimes (\otimes^s T^\star_M)\,. \tag{2.5.1}$$

In our present context, we consider $M := X_0$, that is to say the Zariski open set $X \setminus \mathrm{Supp}(D)$. Let us recall the definition of the *orbifold tensors* introduced by F. Campana [**Cam11a**]. To avoid a possible confusion with the standard orbifold situation (*ie* when $a_i = 1 - \frac{1}{m}$ for some integer m), we will not use his terminology and refer to these tensors as *D-holomorphic tensors.*

Let $x \in X$ be a point; since the hypersurfaces (D_i) have strictly normal intersections, there exist a small open set $\Omega \subset X$, together with a coordinate system $z = (z_1, \ldots, z_n)$ centered at x such that $D_i \cap \Omega = (z_i = 0)$ for $i = 1, \ldots, d$ and $D_i \cap \Omega = \emptyset$ for the others indexes. We define the locally free sheaf $T^r_s(X|D)$ generated as an $\mathcal{O}_X$-module by the tensors

$$z^{\lceil (h_I - h_J)\cdot a \rceil} \frac{\partial}{\partial z_I} \otimes dz^J$$

where the notations are as follows:

1. I (resp. J) is a collection of positive integers in $\{1, \ldots, n\}$ of cardinal r (resp. s) (we notice that we may have repetitions among the elements of I and J, and we count each element according to its multiplicity).
2. For each $1 \leqslant i \leqslant n$, we denote by $h_I(i)$ the multiplicity of i as element of the collection I.
3. For each $i = 1, \ldots, d$ we have $a_i := 1 - \tau_i$, and $a_i = 0$ for $i \geqslant d + 1$.
4. We have
$$z^{\lceil (h_I - h_J)\cdot a \rceil} := \prod_i (z^i)^{\lceil (h_I(i) - h_J(i))\cdot a_i \rceil}$$
5. If $I = (i_1, \ldots, i_r)$, then we have
$$\frac{\partial}{\partial z_I} := \frac{\partial}{\partial z_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial z_{i_r}}$$
and we use similar notations for dz^J.

Hence the holomorphic tensors we are considering here have prescribed zeros/poles near $X \setminus X_0$, according to the multiplicities of D. In the cone case ($D_{lc} = 0$), those tensors have a nice interpretation ([**CGP11**, Lemma 8.2]):

Lemma 2.5.1. — *Assume $D_{lc} = 0$, and let u be a smooth section of the bundle $T_s^r(X_0)$. Then u corresponds to a holomorphic section of $T_s^r(X|D)$ if and only if $\bar\partial u = 0$ and u is bounded with respect to some metric with cone singularities along D.*

In [**CGP11**], the vanishing and parallelism theorems are proved using the classical Bochner formula with an appropriate cut-off function for the space of bounded (for the cone metric) holomorphic sections of $T_s^r(X_0)$, and the lemma above enables to transfer this property to D-holomorphic tensors.
Unfortunately, there is no such simple correspondence in the general log-canonical case. For example, if D has only one component (with coefficient 1) of local equation $z = 0$, then $\frac{dz}{z}$ is a local section of $T_1^0(X|D)$ but it is not bounded with respect to any metric having Poincaré singularities along D.
The idea is to force D-holomorphic tensors to be bounded by twisting them with the trivial line bundle $L = \mathcal{O}_X$ equipped with the singular hermitian metric

$$h_L = e^{-2s\sum_k \log\log\frac{1}{|s_k|^2}} = \prod_{k\in K}\frac{1}{\log^{2s}|s_k|^2}$$

where the $(s_k)_{k\in K}$ are the sections of the divisors D_k appearing in $D_{lc} = \lceil D\rceil$. In more elementary terms, we just change the reference metric measuring those tensors. Then, using a twisted Bochner formula, we will be able to carry on the computations done in [**CGP11**] to obtain the vanishing. It will be practical for the following to introduce the following notation:

Definition 2.5.2. — Let (X, D) be a pair such that D has simple normal crossing support and coefficients in $[0, 1]$. The space of bounded holomorphic tensors of type (r, s) for (X, D) is defined by

$$\mathscr{H}_B^{r,s}(X|D) = \{u \in \mathscr{C}^\infty(X_0, T_s^r(X_0))\,;\, \exists C;\ |u|_h^2 \leqslant C \text{ and } \bar\partial u = 0\}$$

where $h = g_{r,s}\otimes h_L$ is a metric on $T_s^r(X_0)$ induced by h_L and a metric g on X_0 having mixed Poincaré and cone singularities along D.

Of course, this definition does not depend on the choice of the metric g having Poincaré and cone singularities along D; it coincides with the one introduced in [**CGP11**] for klt pairs. The main point about this definition, which legitimates it, consists in the following proposition giving the expected identification between bounded and D-holomorphic tensors:

Proposition 2.5.3. — *With the previous notations, we have a natural identification:*

$$\mathscr{H}_B^{r,s}(X|D) = H^0(X, T_s^r(X|D)).$$

Proof. — We only need to check it locally on $\Omega = (\mathbb{D}^*)^p \times (\mathbb{D}^*)^q \times \mathbb{D}^{n-(k+l)}$, where the boundary divisor restricted to Ω is given by $\sum_{k=1}^p d_k[z_i = 0] + \sum_{k=p+1}^{p+q}[z_k = 0]$, and we choose g to be the model metric ω_D given in the introduction.

Let us begin with the inclusion $\mathscr{H}_B^{r,s}(X|D) \subset H^0(X, T_s^r(X|D))$. By orthogonality of the different $\frac{\partial}{\partial z_I}\otimes dz^J$, we only have to consider $u = v\frac{\partial}{\partial z_I}\otimes dz^J$ for some (holomorphic) function

v satisfying:

$$\frac{|v|}{\prod_{k=1}^{p}|z_k|^{(h_I(k)-h_J(k))a_k}\prod_{k=p+1}^{p+q}|z_k|^{h_I(k)-h_J(k)}\left(\log\frac{1}{|z_k|^2}\right)^{s+h_I(k)-h_J(k)}}\leqslant C$$

Consider now the function

$$w := \frac{v}{\prod_{k=1}^{p} z_k^{\lceil (h_I(k)-h_J(k))a_k\rceil}\prod_{k=p+1}^{p+q} z_k^{h_I(k)-h_J(k)}}$$

whose modulus $|w|$ can also be rewritten in the form

$$\frac{|v|}{\prod_{k=1}^{p}|z_k|^{(h_I(k)-h_J(k))a_k}\prod_{k=p+1}^{p+q}|z_k|^{h_I(k)-h_J(k)}\left(\log\frac{1}{|z_k|^2}\right)^{s+h_I(k)-h_J(k)}}\cdot$$

$$\frac{\prod_{k=p+1}^{p+q}\left(\log\frac{1}{|z_k|^2}\right)^{s+h_I(k)-h_J(k)}}{\prod_{k=1}^{p}|z_k|^{\lceil (h_I(k)-h_J(k))a_k\rceil-(h_I(k)-h_J(k))a_k}}$$

The first factor is bounded; moreover, using the fact that $0\leqslant \lceil x\rceil - x < 1$ for every real number x and that $\left(\log\frac{1}{|z|}\right)^{\alpha}$ is integrable at 0 for every real number α, we conclude that the second factor is also L^2. This finishes to prove that w is L^2, so in particular it extends across the support of our divisor, and therefore, $u\in H^0(\Omega, T_s^r(\Omega|D_{|\Omega}))$.

For the reverse inclusion, every "irreducible" D-holomorphic tensor $u\in H^0(\Omega, T_s^r(\Omega|D_{|\Omega}))$ can be written

$$u=\prod_{k=1}^{p} z_k^{\lceil (h_I(k)-h_J(k))a_k\rceil}\prod_{k=p+1}^{p+q} z^{h_I(k)-h_J(k)} v\frac{\partial}{\partial z_I}\otimes dz^J$$

for some holomorphic function v, and some $I\in\{1,\dots,n\}^r$, $J\in\{1,\dots,n\}^s$. So for g the metric on X_0 attached to ω_D, and setting $h=g_{r,s}\otimes h_L$ as in Definition 2.5.2, we have:

$$|u|_h=\frac{|v|\prod_{k=1}^{p}|z_k|^{\lceil (h_I(k)-h_J(k))a_k\rceil-(h_I(k)-h_J(k))a_k}}{\prod_{k=p+1}^{p+q}\left(\log\frac{1}{|z_k|^2}\right)^{s+h_I(k)-h_J(k)}}$$

which is clearly bounded near the divisor since $s+h_I(k)-h_J(k)\geqslant 0$ for all k. □

Now we can state the main result of this section, which is a partial generalization of [**CGP11**, Theorem C]:

Theorem 2.5.4. — *Let (X,D) be a pair such that $D=\sum a_iD_i$ has simple normal crossing support, with coefficients satisfying: $1/2\leqslant a_i\leqslant 1$ for all i.*
If K_X+D is ample, then there is no non-zero D-holomorphic tensor of type (r,s) whenever $r\geqslant s+1$:

$$H^0(X, T_s^r(X|D))=0.$$

Proof of Theorem 2.5.4. — Proposition 2.5.3 allows us to reduce the vanishing of the D-holomorphic tensors to the one of bounded tensors as defined in 2.5.2. The proof of this result is similar to the one of [**CGP11**, Theorem C], the two main new features being the existence of a Kähler-Einstein metric with mixed Poincaré and cone singularities along D (cf. Theorem A), and the use of a twisted Bochner formula. For this reason, we will give a relatively sketchy proof, and we will refer to [**CGP11**] for the details we skip.

To fix the notations, we write $D = \sum_{j\in J} a_j D_j + \sum_{k\in k} D_k$ where for all $j \in J$, we have $a_j < 1$. In the following, any index j (resp. k) will be implicitely assumed to belong to J (resp. K), whereas the index i will vary in $J \cup K$.
As $K_X + D$ is ample, Theorem A guarantees the existence of a Kähler metric ω_∞ on X_0 such that $-\mathrm{Ric}\,\omega_\infty = \omega_\infty$, and having mixed Poincaré and cone singularities along D. We choose now an element $u \in \mathscr{H}_B^{r,s}(X|D)$ with $r \geqslant s+1$, and we want to use a Bochner formula to show that $u = 0$.
To do this, we need to perform a cut-off procedure, and control the error term so that one can pass to the limit in the cut-off process. Let us now get a bit more into the details.

Step 1: The cut-off procedure
We define $\rho : X \to]-\infty, +\infty]$ by the formula

$$\rho(x) := \log\left(\log \frac{1}{\prod_i |s_i(x)|^2}\right).$$

For each $\varepsilon > 0$, let $\chi_\varepsilon : [0, +\infty[\to [0,1]$ be a smooth function which is equal to zero on the interval $[0, 1/\varepsilon]$, and which is equal to 1 on the interval $[1+1/\varepsilon, +\infty]$. One may for example define $\chi_\varepsilon(x) = \chi_1(x - \frac{1}{\varepsilon})$, so that

$$\sup_{\varepsilon>0, t\in\mathbb{R}_+} |\chi'_\varepsilon(t)| \leqslant C < \infty,$$

and we define $\theta_\varepsilon : X \to [0,1]$ by the expression

$$\theta_\varepsilon(x) = 1 - \chi_\varepsilon\big(\rho(x)\big).$$

We assume from the beginning that we have

$$\prod_i |s_i|^2 \leqslant e^{-2}$$

at each point of X, and then it is clear that we have

$$\theta_\varepsilon = 1 \iff \prod_i |s_i|^2 \geqslant e^{-e^{1/\varepsilon}}$$

and also

$$\theta_\varepsilon = 0 \iff \prod_i |s_i|^2 \leqslant e^{-e^{1+1/\varepsilon}}.$$

We evaluate next the norm of the $(0,1)$–form $\bar\partial\theta_\varepsilon$; we have

$$\bar\partial\theta_\varepsilon(x) = \chi'_\varepsilon\big(\rho(x)\big)\frac{1}{\log\frac{1}{\prod_i |s_i(x)|^2}} \sum_i \frac{\langle s_i, D' s_i\rangle}{|s_i|^2}(x).$$

As ω_∞ has mixed Poincaré and cone singularities along D, we have:

$$|\bar\partial\theta_\varepsilon|^2_{\omega_\infty} \leqslant \frac{C|\chi'_\varepsilon(\rho)|^2}{\log^2\frac{1}{\prod_j |s_j|^2}}\left(\sum_j \frac{1}{|s_j|^{2(1-a_j)}} + \sum_k \log^2|s_k|^2\right) \tag{2.5.2}$$

at each point of X_0. Indeed, this is a consequence of the fact that the norm of the $(1,1)$-forms

$$\frac{i\langle D's_j, D's_j\rangle}{|s_j|^{2a_j}} \quad \text{and} \quad \frac{i\langle D's_k, D's_k\rangle}{|s_k|^2 \log^2 |s_k|^2}$$

with respect to ω_∞ are bounded from above by a constant.

Let $\varepsilon > 0$ be a real number; we consider the tensor

$$u_\varepsilon := \theta_\varepsilon u.$$

It has compact support, hence by the (twisted) Bochner formula (see e.g. [**Dem95**, Lemma 14.2]), we infer

$$\int_{X_0} |\overline{\partial}(\# u_\varepsilon)|_h^2 dV_{\omega_\infty} = \int_{X_0} |\overline{\partial} u_\varepsilon|_h^2 dV_{\omega_\infty} + \int_{X_0} \left(\langle \mathcal{R}(u_\varepsilon), u_\varepsilon\rangle_h + \gamma |u_\varepsilon|_h^2\right) dV_{\omega_\infty} \tag{2.5.3}$$

where:

- $\mathcal{R}$ is a zero-order operator such that in our case ($-\mathrm{Ric}\,\omega_\infty = \omega_\infty$), we have

  $$R_{j\bar{i}} = -\delta_{ji},$$

  and therefore the linear term $\langle \mathcal{R}(u_\varepsilon), u_\varepsilon\rangle$ becomes simply $(s-r)|u_\varepsilon|^2$;

- $h = \omega_{\infty,*} \otimes h_L$, where $\omega_{\infty,*}$ denotes the canonical extension of ω_∞ to the appropriate tensor fields (which are respectively $T_r^s(X_0) \otimes \Omega^{0,1}(X_0)$, $T_s^r(X_0) \otimes \Omega^{0,1}(X_0)$ and $T_s^r(X_0)$);
- $\gamma = \mathrm{tr}_{\omega_\infty}(\Theta_h(L))$ is the trace with respect to ω_∞ of the curvature of (L,h).

Here we need to be cautious because of the singularities of the metric h_L on D. Indeed, the Bochner formula applies to smooth hermitian metrics; howwever one can consider here some metric $h_{L,\varepsilon}$ which would coincide with h_L whenever $\theta_\varepsilon > 0$ and which is a smooth metric near D. For example, on can set $h_{L,\varepsilon} = \theta_{\varepsilon/2} h_L + (1-\theta_{\varepsilon/2})$. Then for each $\varepsilon < 1$, there exists an open set $U_\varepsilon \supset \overline{\{\theta_\varepsilon > 0\}}$ on which $h_{L,\varepsilon} = h_L$ so that in particular, in the formula (2.5.3), one can replace h_L by $h_{L,\varepsilon}$ without affecting anything.

There remains two steps to achieve now: the first one consists in evaluating the correction term γ induced by the curvature of L, and the second one is to show that the integration by part is valid in the Poincaré-cone setting; more precisely we have to prove that the error term $\int_{X_0} |\bar{\partial} u_\varepsilon|_h^2 dV_{\omega_\infty}$ converges to 0 as ε goes to 0.

Step 2: Dealing with the curvature of (L,h)
We work on local charts where D_{lc} is given by $\{\prod_{k\in K} z_k = 0\}$.
To begin with, we know that there exists $A > 0$ such that $\omega_\infty \leqslant A\left(\omega_{\mathrm{klt}} + \sum_k \frac{idz_k \wedge d\bar{z}_k}{|z_k|^2 \log^2 |z_k|^2}\right)$ where ω_{klt} is some smooth metric on $X \setminus \mathrm{Supp}(D_{klt})$ having cone singularities along D_{klt}. It will be useful to introduce the notation $\omega_{\mathrm{lc}} := \omega_{\mathrm{klt}} + \sum_k \frac{idz_k \wedge d\bar{z}_k}{|z_k|^2 \log^2 |z_k|^2}$. Moreover, the usual

computations (see e.g.[**Kob84**, Lemma 1]) show that there exists a smooth $(1,1)$-form α on our chart satisfying

$$-\sum_{k\in K} dd^c \log\log \frac{1}{|s_k|^2} \geqslant \sum_{k\in K} \frac{idz_k \wedge d\bar{z}_k}{|z_k|^2 \log^2 |z_k|^2} + \frac{1}{B}\alpha$$

where B is a constant which can be taken as large as wanted up to scaling the (smooth) metrics on the D_k's, which does not affect their curvature. Therefore, the curvature $\Theta_{h_L}(L)$ of L satisfies:

$$\begin{aligned} \mathrm{tr}_{\omega_\infty}(-\Theta_{h_L}(L)) &\geqslant A^{-1}\mathrm{tr}_{\omega_{\mathrm{lc}}}(-\Theta_{h_L}(L)) \\ &\geqslant 2sA^{-1}\mathrm{tr}_{\omega_{\mathrm{lc}}}\left(\sum_{k\in K} \frac{idz_k \wedge d\bar{z}_k}{|z_k|^2 \log^2 |z_k|^2} + \frac{1}{B}\alpha\right) \\ &\geqslant 2s|K|A^{-1} + 2s(AB)^{-1}\mathrm{tr}_{\omega_{\mathrm{lc}}}\alpha \end{aligned}$$

As ω_{lc} dominates some smooth form on X, the quantity $\mathrm{tr}_{\omega_{\mathrm{lc}}}\alpha$ is bounded on X_0 so that $2s(AB)^{-1}\mathrm{tr}_{\omega_{\mathrm{lc}}}\alpha$ can be made as small as we want by scaling the metrics on the divisors as explained above. Therefore one has

$$\gamma = \mathrm{tr}_{\omega_\infty}(\Theta_{h_L}(L)) \leqslant \frac{1}{2} \tag{2.5.4}$$

on X_0.

Step 3: Controlling the error term

Let us get now to the last step in showing that the term

$$\int_{X_0} |\bar{\partial} u_\varepsilon|_h^2 dV_{\omega_\infty}$$

tends to zero as $\varepsilon \to 0$. Since u is holomorphic, we have

$$\bar{\partial} u_\varepsilon = u \otimes \bar{\partial}\theta_\varepsilon;$$

we recall now that $u \in \mathscr{H}_B^{r,s}(X|D)$, so we have

$$|\bar{\partial} u_\varepsilon|_h^2 \leqslant C|\bar{\partial}\theta_\varepsilon|_{\omega_\infty}^2. \tag{2.5.5}$$

By inequality (2.5.2) above we infer

$$\int_{X_0} |\bar{\partial} u_\varepsilon|_h^2 dV_{\omega_\infty} \leqslant C\int_{X_0} \frac{|\chi_\varepsilon'(\rho)|^2}{\log^2 \frac{1}{\prod_i |s_i|^2}} \left(\sum_j \frac{1}{|s_j|^{2(1-a_j)}} + \sum_k \log^2 |s_k|^2\right) dV_{\omega_\infty}. \tag{2.5.6}$$

As ω_∞ as mixed Poincaré and cone singularities along D, we have:

$$\int_{X_0} |\bar{\partial} u_\varepsilon|_h^2 dV_{\omega_\infty} \leqslant C\int_{X_0} \frac{|\chi_\varepsilon'(\rho)|^2 \left(\sum_j \frac{1}{|s_j|^{2(1-a_j)}} + \sum_k \log^2 |s_k|^2\right)}{\prod_j |s_j|^{2a_j} \prod_k |s_k|^2 \log^2 |s_k|^2 \cdot \log^2 \frac{1}{\prod_i |s_i|^2}} dV_\omega. \tag{2.5.7}$$

for some constant $C > 0$ independent of ε; here we denote by ω a smooth hermitian metric on X. We remark that the support of the function $\chi'_\varepsilon(\rho)$ is contained in the set

$$e^{-e^{1+1/\varepsilon}} \leqslant \prod_i |s_i|^2 \leqslant e^{-e^{1/\varepsilon}}$$

so in particular we have

$$\frac{|\chi'_\varepsilon(\rho)|^2}{\log^{\frac{1}{2}} \frac{1}{\prod_j |s_i|^2}} \leqslant Ce^{-\frac{1}{2\varepsilon}}. \tag{2.5.8}$$

We also notice that for each indexes $j_0 \in J$ and $k_0 \in K$ we have respectively:

$$\int_{X_0} \frac{dV_\omega}{|s_{j_0}|^2 \log^{3/2}\left(\frac{1}{\prod_i |s_i|^2}\right) \prod_{j \neq j_0} |s_j|^{2a_j} \prod_k |s_k|^2 \log^2 |s_k|^2}$$
$$\leqslant C \int_{X_0} \frac{dV_\omega}{|s_{j_0}|^2 \log^{3/2}\left(\frac{1}{|s_{j_0}|^2}\right) \prod_{j \neq j_0} |s_j|^{2a_j} \prod_k |s_k|^2 \log^2 |s_k|^2}$$

and

$$\int_{X_0} \frac{dV_\omega}{|s_{k_0}|^2 \log^{3/2}\left(\frac{1}{\prod_i |s_i|^2}\right) \prod_j |s_j|^{2a_j} \prod_{k \neq k_0} |s_k|^2 \log^2 |s_k|^2}$$
$$\leqslant C \int_{X_0} \frac{dV_\omega}{|s_{k_0}|^2 \log^{3/2}\left(\frac{1}{|s_{k_0}|^2}\right) \prod_j |s_j|^{2a_j} \prod_{k \neq k_0} |s_k|^2 \log^2 |s_k|^2}$$

and the integral in the right hand sides are convergent, given that the hypersurfaces (D_i) have strictly normal intersections.
Finally we combine the inequalities (2.5.7)-(2.5.8), and we get

$$\int_{X_0} |\bar\partial u_\varepsilon|^2 dV_{\omega_\infty} \leqslant Ce^{-\frac{1}{2\varepsilon}}. \tag{2.5.9}$$

Step 4: Conclusion
As we can see, the relations (2.5.3) and (2.5.9) combined with the fact, coming from (5.3.12), that

$$\langle \mathcal{R}(u_\varepsilon), u_\varepsilon \rangle_h + \gamma |u_\varepsilon|_h^2 \leqslant \left(\frac{1}{2} + s - r\right) |u_\varepsilon|_h^2$$

(which tends to $(\frac{1}{2} + s - r)|u|_h^2)$ will give a contradiction if u is not identically zero on X_0 (we recall that by hypothesis we have $r \geqslant s + 1$).

□

CHAPITRE 3

KÄHLER-EINSTEIN METRICS WITH CONE SINGULARITIES ON KLT PAIRS

Dans ce chapitre, on va approfondir les résultats du chapitre précédent – au moins pour la partie conique– au cas des variétés singulières. La généralisation de la notion de paire (X, D) composée d'une variété projective lisse X et d'un diviseur effectif D à coefficients dans $]0, 1[$, et dont le support soit à croisements normaux, est celle de paire klt. Il s'agit alors toujours d'une paire (X, D), mais cette fois-ci, X peut être singulière, et D est seulement un diviseur de Weil (effectif disons). La condition sur les coefficients se transforme alors en une condition analogue exprimée dans une log résolution de la paire.

Il existe alors une notion de métrique de Kähler-Einstein pour de telles paires, mais en un sens assez faible. Par exemple, on sait que ces métriques sont lisses sur la partie régulière de X privée de D, mais on ne connait pas vraiment leur comportement ailleurs. Le but du chapitre qui suit est de voir (au moins sous une hypothèse technique) que sur l'ouvert de Zariski où la paire est log lisse, alors toute métrique KE (quelle que soit la courbure) est bien à singularités coniques le long de D, au sens habituel.

3.1. Introduction

Let (X, D) be a Kawamata log-terminal pair (shortened in *klt*), i.e. X is a normal projective variety over $\mathbb{C}$ of dimension n, and D is an arbitrary $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier, and for some (or equivalently any) log-resolution $\pi : X' \to X$, we have:

$$K_{X'} = \pi^*(K_X + D) + \sum a_i E_i$$

where E_i are either exceptional divisors or components of the strict transform of D, and the coefficients a_i satisfy the inequality $a_i > -1$.

For such a pair (X, D), there exists a natural notion of Kähler-Einstein metric developed by Eyssidieux, Guedj and Zeriahi in [**EGZ09**] for the non-positively curved case (ie $K_X + D$ ample or trivial), and extended by Berman, Boucksom and the previous authors in [**BBE$^+$11**] for log-Fano varieties ($-(K_X + D)$ ample). Moreover, if $K_X + D$ is merely big, there exists also a unique Kähler-Einstein metric thanks to the finite generation of the ring $\oplus_{m \geqslant 0} H^0(X, m(K_X + D))$ proved by [**BCHM10**], cf. e.g. [**BEGZ10**, section 6] or the explanations below Theorem A. In that case, there is a particular Zariski open subset of

X introduced by Boucksom in [**Bou04**] called ample locus and denoted by $\mathrm{Amp}(K_X + D)$ which plays a special role : it is an open subset where the cohomology class $c_1(K_X + D)$ "looks like" a Kähler class. We refer to section 3.2.1.1 for the definitions of those objects.

Those objects are currents living on a log-resolution of the pairs, and are very particular from the point of view of pluripotential theory (they have finite energy). Moreover, those currents are shown to induce genuine Kähler-Einstein metrics on the Zariski open subset $X_0 := X_{\mathrm{reg}} \setminus \mathrm{Supp}(D)$. However, little is known on the behavior of the Kähler-Einstein metrics near the degeneracy locus $X \setminus X_0$.

As understanding the degeneracy of the Kähler-Einstein metric near the singularities of X seems out reach for the moment, we will focus on the behavior near the support of D intersected with the regular locus of X. It turns out that this is equivalent to control the singularities of the solution of a degenerate Monge-Ampère equation on a *smooth manifold*, the term "degenerate" meaning here that the solution lives in a class which is not Kähler (but at least big, or nef and big), and that the right-hand side is not smooth (cf Theorem B). To avoid the singularities of the pair (X, D), we will restrict to study the Kähler-Einstein metric on the open subset of X which is not affected by the log-resolution: this is the set of points x where (X, D) is log-smooth at x, in the sense that x is a smooth point of X, and there is a (small) analytic neighborhood U of x such that $\mathrm{Supp}(D) \cap U = \{z_1 \cdots z_k = 0\}$ for some holomorphic coordinates $z_1, \ldots, z_n$.

Theorem A. — *Let (X, D) be a klt pair, and let $\mathrm{LS}(X, D) := \{x \in X; (X, D)$ is log-smooth at $x\}$. We assume that the coefficients of D are in $[1/2, 1)$.*

- (i) *If $K_X + D$ is big, then the Kähler-Einstein metric of (X, D) has cone singularities along D on $\mathrm{LS}(X, D) \cap \mathrm{Amp}(K_X + D)$.*
- (ii) *If $-(K_X + D)$ is ample, then any Kähler-Einstein metric of (X, D) has cone singularities along D on $\mathrm{LS}(X, D)$.*

In order to relate this result to previous works, we should mention that a result like Theorem A was already known when (X, D) is a log-smooth pair with $K_X + D$ ample. More precisely, this result was obtained in [**Bre11, JMR11**] whenever D is irreducible and in [**CGP11**] when the coefficients of D are in $[1/2, 1)$. Moreover, the general case has been announced by R. Mazzeo and Y. Rubinstein in [**MR12**]. The assumption on the coefficients of the divisor comes from the regularization technique of the cone metric in [**CGP11**], which yields a Kähler metric with curvature (uniformly) bounded below only when the coefficients are greater than $1/2$, and as we really need a control on the curvature along the process, we have to make that assumption.

Let us mention that we also prove a similar result for the Ricci-flat case : if under the same set-up, we are given a nef and big class α, and assume that $c_1(K_X + D)$ is trivial, then the Ricci-flat metric in α has cone singularities along D on $\mathrm{LS}(X, D) \cap \mathrm{Amp}(K_X + D)$. As the result is a bit less natural, we chose not to include it in the previous Theorem.

We should also mention that $\mathrm{LS}(X, D)$ is a Zariski open subset of X with complement of codimension at least 2, as it contains the intersection of X_{reg} with the regular locus of D_{red}.

Let us explain case (i) in Theorem A. For the moment, we do not know any regularity result for solutions of Monge-Ampère equations in general big classes. But in the case of adjoint bundles coming from klt pairs, the main theorems of the Minimal Model Program (MMP) theory enable us to reduce to the semipositive and big case which is far better understood. Let us now be more precise:

Let (X, D) be a klt pair of log-general type, ie $K_X + D$ is assumed to be big. By the fundamental results of [**BCHM10**, Theorem 1.2], we know that (X, D) has a log-canonical model $f : X \dashrightarrow X_{\mathrm{can}}$ where X_{can} is a projective normal variety and f is a birational contraction such that $f_*(K_X + D) = K_{X_{\mathrm{can}}} + D_{\mathrm{can}}$ is ample (here $D_{\mathrm{can}} := f_* D$). If $\mu : Y \to X$, $\nu : Y \to X_{\mathrm{can}}$ is a resolution of the graph of f, then

$$\mu^*(K_X + D) = \nu^*(K_{X_{\mathrm{can}}} + D_{\mathrm{can}}) + E$$

for some effective ν-exceptional divisor E. A consequence of this Zariski decomposition is that every positive current T in $c_1(K_X + D)$ comes from a unique positive current $S \in c_1(K_{X_{\mathrm{can}}} + D_{\mathrm{can}})$ in the following way: $\mu^* T = \nu^* S + [E]$. In particular the Kähler-Einstein metric on (X, D) constructed in [**BEGZ10**, Theorem 6.4] (see also Theorem 3.2.2 for the general case of a klt pair) corresponds in this way to the Kähler-Einstein metric on $(X_{\mathrm{can}}, D_{\mathrm{can}})$ constructed in [**EGZ09**, Theorem 7.12]. In particular, this last metric is smooth outside of the singular locus of X_{can} and $\mathrm{Supp}(D_{\mathrm{can}})$.

Moreover, as the log-canonical model $f : X \dashrightarrow X_{\mathrm{can}}$ induces an isomorphism from the ample locus of $K_X + D$ onto its image (this is a general property for the maps attached to big linear systems, which follows directly from the definition of $\mathbb{B}_+(K_X + D)$), it is enough for our matter to understand the Kähler-Einstein metric of $(X_{\mathrm{can}}, D_{\mathrm{can}})$. So we are reduced to working in semipositive and big classes as explained above.

Once this reduction is done, we can express the problem in terms of Monge-Ampère equations (cf section 3.3.1); the framework is the following one: let X be a compact Kähler manifold of dimension n, α a nef and big class class and $\theta \in \alpha$ a smooth representative. Let $E = \sum c_j E_j$ be an effective $\mathbb{R}$-divisor with snc support such that $\alpha - E$ is Kähler. Let also $D = \sum a_i D_i$ be an effective divisor with snc support such that E and D have no common components, and that $E + D$ has snc support.

We write $X_0 = X \setminus (\mathrm{Supp}(E) \cup \mathrm{Supp}(D))$, we choose non-zero global sections t_j of $\mathcal{O}_X(E_j)$ and s_i of $\mathcal{O}_X(D_i)$, and we choose some real numbers $b_j > -1$. We choose some smooth hermitian metrics on those bundles which we normalize so that $\int_X \prod |t_j|^{2b_j} \prod |s_i|^{-2a_i} dV = \mathrm{vol}(\alpha)$. Finally, if φ is a θ-psh function, we denote by $\mathrm{MA}(\varphi)$ the non-pluripolar product $\langle (\theta + dd^c \varphi)^n \rangle$ in the sense of [**BEGZ10**], cf. section 3.2.2.

Theorem B. — *We assume that the coefficients of D satisfy the inequalities $a_i \geqslant 1/2$ for all i. Then any solution with full Monge-Ampère mass of*

$$\mathrm{MA}(\varphi) = \prod |t_j|^{2b_j} \frac{e^{\lambda\varphi} dV}{\prod |s_i|^{2a_i}}$$

defines a smooth metric on X_0 having cone singularities along D.

Let us mention that if $\lambda \geqslant 0$, such a solution always exists and is unique. If $\lambda < 0$, there might be no solution, or on the contrary, many ones. We recall (cf comments below Theorem A) that the assumptions on the coefficients of the divisor are made to ensure that we deal with approximate cone metrics having curvature uniformly bounded *below*, cf also [**CGP11**] where the same difficulty appears.

In order to deduce Theorem A from Theorem B, one considers a log-resolution of the pair (X, D) which also computes the augmented base locus as an SNC divisor meeting the strict transform of D properly. The Monge-Ampère equation giving the Kähler-Einstein metrics pulls back to this resolution and has exactly the form considered above. Finally, we observe that a log-resolution of a pair (X, D) is an isomorphism on the previously defined log-smooth locus $\mathrm{LS}(X, D)$, so we are done.

One may remark that to deduce Theorem A from Theorem B, it would have been enough to assume α semi-positive and big. As the proof of the nef and big case is not really more complicated, we chose to state the Theorem in this slightly greater generality. We should also add that these last results are expected to be valid in the more general case where α is only big and not necessarily nef. However, even when $D = 0$, we were not able to prove that the Kähler-Einstein metrics are smooth on the ample locus of α, and new ideas shall probably be needed to settle this question.

In fact, using the same techniques appearing in the proof of Theorem B, we can formulate a slightly more general result (cf. Theorem 3.4.4): from the Monge-Ampère point of view (Theorem B), we do not need that the factors $|t_j|^{-2b_j}$ come from the augmented base locus E. Therefore, even if D has coefficients in $(0, 1)$ and not only $[1/2, 1)$, one can still prove that the Kähler-Einstein metrics in Theorem A will have cone singularities along $\sum_{a_i \geqslant 1/2} D_i$ when restricted to $X_{\mathrm{reg}} \cap \mathrm{LS}(X, D) \setminus \mathrm{Supp}(\sum_{a_i < 1/2} D_i)$ (and intersected with $\mathrm{Amp}(K_X + D)$ in case (i)).

Finally, let us say a word about the new difficulties (in relation to [**CGP11**] appearing in Theorem B from the singular setting. A first problem is that we a priori do not have at our disposal a sequence of approximate cone metric ω_ε on the ample locus having bisectional curvature uniformly bounded from below. Indeed, we do have a smooth Kähler metric ω on the ample locus Ω which extends *on a compactification* of Ω, but it is not clear at all that the metrics ω_ε obtained from ω by adding a approximate cone potential $dd^c\psi_\varepsilon$ as in [**CGP11**] will have bisectional curvature bounded from below.
Therefore we do need to blow-up X and consider a log-resolution of the augmented base locus. But then, upstairs, the Monge-Ampère equation acquires poles and zeros along the

exceptional divisor, and the strategy of [**CGP11**] cannot apply directly.
The key point is to use (and refine slighlty) an estimate appearing formerly in [**Pău08**] and explicited in the recent article [**BBE$^+$11**]. This estimate was used in particular to show the smoothness of weak Kähler-Einstein metrics on Fano manifolds, giving then a different proof than [**ST11**] where the strategy consisted in exploiting the regularizing properties of the Kähler-Ricci flow.

3.2. Monge-Ampère equations in big cohomology classes

In this section, we recall some generalities on big cohomology classes on a compact Kähler manifold, and then give an outline of the paper by Boucksom, Eyssidieux, Guedj and Zeriahi [**BEGZ10**] which we are going to rely on.

3.2.1. Generalities on big cohomology classes. — We start with a compact Kähler manifold X of dimension n, and we consider a class $\alpha \in H^{1,1}(X, \mathbb{R})$ which is big. By definition, this means that α lies in the interior of the pseudo-effective cone, so that there exists a Kähler current $T \in \alpha$, that is a current which dominates some smooth positive form ω on X.

3.2.1.1. The ample locus of α. — One may define, following S. Boucksom [**Bou04**, §3.5], the *ample locus* of α, denoted $\mathrm{Amp}(\alpha)$, which is the largest Zariski open subset U of X such that for all $x \in U$, there exists a Kähler current $T_x \in \alpha$ with analytic singularities such that T_x is smooth in an (analytic) neighbourhood of x. A generalization of Kodaira's lemma asserts that for some modification $\pi : X' \to X$, one can write $\pi^*\alpha = \beta + E$ where β is a Kähler class and E is an effective divisor on X'. Then one has the following characterization of the complement of $\mathrm{Amp}(\alpha)$, denoted $\mathbb{B}_+(\alpha)$ and called *augmented base locus* in analogy with the case where α is a big class in the real Néron-Severi group $NS(X) \otimes \mathbb{R}$ (cf [**ELM$^+$06**], [**BBP10**, Lemma 1.4]):

$$\mathbb{B}_+ = \bigcap_{\pi^*\alpha - E \text{ Kähler}} \pi(\mathrm{Supp}(E))$$

where E ranges over all effective $\mathbb{R}$-divisor in a birational model $\pi : X' \to X$ such that $\pi^*\alpha - E$ is a Kähler class.

At this point, two remarks need to be made. The first one is that the augmented base locus (or equivalently the ample locus) of a big class α can be computed by a single modification $\pi : X' \to X$.
Indeed, by the noetherianity of X for the (holomorphic) Zariski topology, there exists a Kähler current $T \in \alpha$ with analytic singularities such that the singular locus of T is exactly $\mathbb{B}_+$. Resolving the singularities of T (cf [**Bou04**, §2.6]), one obtains a morphism $\pi : X' \to X$ such that $\pi^*T = \theta + [E]$ where $\theta \geqslant \pi^*\omega$ for any Kähler form ω on X dominated by T, and $E = \sum a_i E_i$ is an effective $\mathbb{R}$-divisor lying above the singular locus of T (so it is not necessarily exceptional because $\mathbb{B}_+$ might have one-codimensional components). Moreover, as π is a birational morphism between smooth varieties (actually we use here that X is locally $\mathbb{Q}$-factorial) there exists some positive linear combination of exceptional divisors $F = \sum b_i E_i$ such that $-F$ is π-ample (cf [**Har77**, II, ex. 7.11 (c)] or [**Deb01**, 1.42]). Therefore, for a

sufficiently small ε, the cohomology class of $\theta - \varepsilon F$ contains a Kähler form, so that we have the following decomposition:

$$\pi^*\alpha = \{\theta - \varepsilon F\} + (E + \varepsilon F)$$

with $\{\theta - \varepsilon F\}$ Kähler, and $E + \varepsilon F$ effective, with support equal to $\mathrm{Supp}(E)$. Therefore $\mathbb{B}_+ = \pi(\mathrm{Supp}(E))$, which shows that $\mathbb{B}_+$ can be indeed computed by a single modification of X.

The second remark we would like to do about the notion of ample locus concerns the case when $\alpha = c_1(L)$ is the Chern class of a line bundle. In that case, there is no need to perform modifications of X to compute $\mathbb{B}_+$, as is shown in [**ELM$^+$06**, Remark 1.3]:

$$\mathbb{B}_+(L) = \bigcap_{\substack{L=A+E \\ A \text{ ample},\, E\geqslant 0}} \mathrm{Supp}(E)$$

3.2.1.2. Currents with minimal singularities. — We will be very brief about this well-known notion, and refer e.g. to [**Bou04**, §2.8], [**BBGZ09**, §1], [**Ber09**] or [**BD12**] for more details and recent results.

By definition, if T, T' are two positive closed currents in the same cohomology class α, we say that T is less singular than T' if the local potentials φ, φ' of these currents satisfy $\varphi' \leqslant \varphi + O(1)$. It is clear that this definition does not depend on the choice of the local potentials, so that the definition is coherent. In each (pseudo-effective) cohomology class α, one can find a positive closed current $T_{\min}$ which will be less singular than all the other ones; this current is not unique in general; only its class of singularities is. Such a current will be called current with minimal singularities.

One way to find such a current is to pick $\theta \in \alpha$ a smooth representative, and define then, following Demailly, the upper envelope

$$V_\theta := \sup\{\varphi\ \theta\text{−psh},\ \varphi \leqslant 0 \text{ on } X\}$$

Once observed that V_θ is θ-psh (in particular upper semi-continuous), it becomes clear that $\theta + dd^c V_\theta$ has minimal singularities.

3.2.2. Non-pluripolar product and Monge-Ampère equations. —

3.2.2.1. Non-pluripolar product. — In the paper [**BEGZ10**], the *non-pluripolar product* $T \mapsto \langle T^n \rangle$ of any closed positive $(1,1)$-current $T \in \alpha$ is shown to be a well-defined measure on X putting no mass on pluripolar sets. Given now a θ-psh function φ, one defines its non-pluripolar Monge-Ampère by $\mathrm{MA}(\varphi) := \langle (\theta + dd^c\varphi)^n \rangle$. Then one can check easily from the construction that the total mass of $\mathrm{MA}(\varphi)$ is less than or equal to the volume $\mathrm{vol}(\alpha)$ of the class α:

$$\int_X \mathrm{MA}(\varphi) \leqslant \mathrm{vol}(\alpha)$$

A particular class of θ-psh functions that appears naturally is the one for which the last inequality is an equality. We will say that such functions (or the associated currents) have *full Monge-Ampère mass*.

Let us consider now the case of θ-psh functions with minimal singularities. By definition, they are locally bounded on $\mathrm{Amp}(\alpha)$, so that one can consider on this open set their Monge-Ampère $(\theta + dd^c\varphi)^n$ in the usual sense of Bedford-Taylor. Then one can see that the trivial extension of this measure to X coincides with $\mathrm{MA}(\varphi)$ and satisfies

$$\int_X \mathrm{MA}(\varphi) = \mathrm{vol}(\alpha)$$

In particular, currents with minimal singularities have full Monge-Ampère mass, the converse being false however. An observation that dates back to S. Boucksom [**Bou04**, Proposition 3.6] shows that whenever α is nef and big, then the positive currents in α having minimal singularities automatically have zero Lelong numbers, or equivalently, using Skoda's integrability theorem [**Sko72**], their potentials φ satisfy $e^{-\varphi} \in L^p$ for all $p \geqslant 1$.

Very recently, a similar statement has been obtained for semi-positive and big classes by Berman, Boucksom, Eyssidieux, Guedj and Zeriahi. The precise statement is the following one:

Theorem 3.2.1 ([**BBE$^+$11**, Theorem 1.1]). — *Let X be a normal compact complex space endowed with a fixed Kähler form ω_0. Let φ be an ω_0-psh function with full Monge-Ampère mass, and $\pi : X' \to X$ be any resolution of singularities of X. Then $\varphi' := \varphi \circ \pi$ has zero Lelong numbers everywhere. Equivalently, $e^{-\varphi'} \in L^p(X')$ for all $p \geqslant 1$.*

This result will be very helpful for the proof of Theorem B. If we did not have it, then in the case $\lambda < 0$, we should have added the assumption that φ has minimal singularities.

3.2.2.2. Monge-Ampère equations in big cohomology classes. — One of the main results of the paper [**BEGZ10**], is that for every non-pluripolar measure μ, there exists a unique positive current $T_\mu \in \alpha$ with full Monge-Ampère mass satisfying the Monge-Ampère equation

$$\langle T^n \rangle = \mu$$

The strategy of the proof is to consider approximate Zariski decompositions $X_k \xrightarrow{\pi_k} X$ (with $\pi_k^*\alpha = \beta_k + [E_k]$ where β_k is Kähler and E_k effective) and to solve $\langle S_k^n \rangle = \frac{\mathrm{vol}(\beta_k)}{\mathrm{vol}(\alpha)} \pi_k^* \mu$ with $S_k \in \beta_k$, which is possible thanks to the main result of [**GZ07**] (β_k is Kähler). Then one needs to prove that $T_k := (\pi_k)_*(S_k + [E_k])$ converges to some current T with full Monge-Ampère mass solution of the initial equation.

In that same paper, the authors of [**BEGZ10**] obtain L^∞ estimates of the potential of the solution T whenever the measure $\mu = f dV$ has $L^{1+\varepsilon}$-density with respect to the Lebesgue measure. More precisely, they get the following result [**BEGZ10**, Theorem 4.1]: the normalized potential φ (ie $\max_X \varphi = 0$) solution of $\mathrm{MA}(\varphi) = \mu$ satisfies

$$\varphi \geqslant V_\theta - M ||f||_{L^{1+\varepsilon}}^{1/n}$$

where V_θ is the upper envelope $V_\theta := \sup\{\psi \ \theta\text{−psh}, \psi \leqslant 0 \text{ on } X\}$ defined in the previous section, and M depends only on θ, dV and ε.

Finally, if $\mu = dV$ is now a smooth volume form, and under the additional assumption that α is nef and big, then [**BEGZ10**, Theorem 5.1] asserts that the solution T_μ is smooth on the ample locus $\mathrm{Amp}(\alpha)$. Very little is known however about the behavior of T_μ along $\mathbb{B}_+$, even in the case where α is semi-positive (and big).

3.2.3. The equation $\mathrm{MA}(\varphi) = e^\varphi \mu$**.** — In this section, we focus on the equation $\mathrm{MA}(\varphi) = e^\varphi \mu$, where μ is a non-pluripolar measure. As we explained in the introduction, this equation is related to negatively curved Kähler-Einstein metrics.

Whenever $\mu = dV$ is a smooth volume form, we have at our disposal [**BEGZ10**, Theorem 6.1] which garantees that the previous equation admits a unique solution φ θ-psh with full Monge-Ampère mass (we are still assuming that $\alpha = \{\theta\}$ is a big class).
In fact, their proof can be readily adapted to the case where μ has a $L^{1+\varepsilon}$ density with respect to the Lebesgue measure:

Theorem 3.2.2. — *Let X be a compact Kähler manifold of dimension n, $\alpha \in H^{1,1}(X,\mathbb{R})$ a big class, and $\mu = f dV$ a volume form with density $f \geqslant 0$ belonging to $L^{1+\varepsilon}(dV)$ for some $\varepsilon > 0$. Then there exists a unique θ-psh function φ with full Monge-Ampère mass such that $\langle(\theta + dd^c\varphi)^n\rangle = e^\varphi \mu$. Furthermore, φ has minimal singularities.*

Sketch of Proof. — The proof is almost the same as the one of [**BEGZ10**, Theorem 6.1], so we only give the main ideas. We may assume without loss of generality that μ has total mass 1, and consider $\mathcal{C}$ the subset of $L^1(X,\mu)$ consisting of all θ-psh functions ψ normalized by $\sup_X \psi = 0$. Indeed, $PSH(X,\theta) \subset L^1(X,\mu)$ because any θ-psh function is in L^p_{loc} for every $p > 0$, so applying Hölder's inequality with $p = 1 + \frac{1}{\varepsilon}$, we obtain the result. This set is convex, compact, so there exists $C > 0$ such that $\int_X \psi d\mu \geqslant -C$ for all $\psi \in \mathcal{C}$, as explained in [**GZ05**, Proposition 1.7]. By convexity, one deduces in particular that $\log \int_X e^\psi d\mu \geqslant -C$ for all $\psi \in \mathcal{C}$.

For all $\psi \in \mathcal{C}$, the measure $e^\psi \mu$ has uniform $L^{1+\varepsilon}$ density with respect to dV because $\psi \leqslant 0$. Therefore, [**BEGZ10**, Theorems 3.1 & 4.1] ensure the existence of a unique function $\Phi(\psi) \in \mathcal{C}$ such that $\mathrm{MA}(\Phi(\psi)) = e^{\psi + c_\psi}\mu$, where $c_\psi = \log \mathrm{vol}(\alpha) - \log \int_X e^\psi d\mu \leqslant \log \mathrm{vol}(\alpha) + C$, and $\Phi(\psi) \geqslant V_\theta - M$ for some uniform M.

[**BEGZ10**, Lemma 6.2] shows that the map $\Phi : \mathcal{C} \to \mathcal{C}$ is continuous, therefore it has a fixed point $\psi \in \mathcal{C}$ by Schauder's fixed point theorem, and one concludes setting $\varphi := \psi + c_\psi$. As any θ-psh function is bounded above on X, every solution of our equation has minimal singularities, and thus uniqueness follows from the straightforward generalization of [**BEGZ10**, Proposition 6.3] in the setting of measures with $L^{1+\varepsilon}$ density with respect to the Lebesgue measure. □

Remark 3.2.3. — Let us note that it follows from the proof of this theorem that there exists M depending only on θ, dV and ε such that the solution φ of $\langle(\theta + dd^c\varphi)^n\rangle = e^\varphi \mu$ satisfies $M \geqslant \varphi \geqslant V_\theta - M||f||_{L^{1+\varepsilon}}^{1/n}$. Indeed, the only point is to control the constant C appearing in the previous proof, but C is bounded by $\sup\{||\psi||_{L^{1+1/\varepsilon}} \cdot ||f||_{L^{1+\varepsilon}}\, ; \psi \in \mathcal{C}\}$ which is finite by compactness of $\mathcal{C}$ and equivalence of the L^1 and L^p topology for quasi-psh functions.

3.3. Cone singularities for Kähler-Einstein metrics

3.3.1. Singular Kähler-Einstein metrics. — As we mentioned in the introduction, one can define the notion of Kähler-Einstein metric attached to any klt pair (X, D). We refer e.g. to [**EGZ09**] or [**BBE**$^+$**11**] for the definition in the non-positively curved case and in the log-Fano case respectively. What is important to remember about those objects is that they are currents on the singular variety X which satisfy on any log-resolution (X', D') (D' being given by the identity $K_{X'} + D' = \pi^*(K_X + D)$) a Monge-Ampère equation of the form

$$\text{(KE)} \qquad \text{MA}(\phi) = e^{\pm(\phi - \phi_{D'})}$$

in the negatively or positively curved case (and a similar equation in the Ricci-flat case). Here ϕ is a (singular) psh weight on $\pm c_1(K_{X'} + D')$ (or in some given semiample and big class), $\phi_{D'}$ is a (singular) weight on the $\mathbb{R}$-line bundle $\mathcal{O}_X(D')$ such that $dd^c\phi_{D'} = [D']$, and MA is the non-pluripolar Monge-Ampère operator.

A Kähler-Einstein metric $\omega = dd^c\phi$ attached to (X, D) shall satisfy the equation

$$\operatorname{Ric}\omega = \mp\omega + [D]$$

(whenever $\operatorname{Ric}\omega = -dd^c \log \omega^n$ makes sense) or equivalently $\operatorname{Ric}\omega' = \mp\omega' + [D']$ where $\omega' = \pi^*\omega$ for a log-resolution $\pi : X' \to X$ of (X, D).

It is not completely clear that any Kähler-Einstein metric (as previously defined) should be smooth on $X_{\text{reg}} \setminus \text{Supp}(D)$, or equivalently on $X' \setminus \text{Supp}(D')$. This work has been done for the log-Fano case in [**BBE**$^+$**11**] using an estimate appearing in [**Pău08**]. We will also follow this strategy to obtain the smoothness on the suitable locus (cf Remark 3.4.2). This strategy will also enable us to establish the cone singularities of the Kähler-Einstein metric).

There are two difficulties arising when one wants to understand the solutions of the Monge-Ampère equation (KE_λ): first of all, the right-hand side is singular, and also the class in which one looks for a solution is no longer ample (or Kähler) but merely semi-ample and big. However, working with those weak positivity notions has the advantage that our equation is invariant by modification, and can be read equally on any log-resolution. Therefore, one can assume that the augmented base locus of $\pm\pi^*(K_X + D)$ (or $\pi^*\alpha$ for α a Kähler class on X, in the Ricci-flat case) is given by a divisor E (in a sense to make clear) having simple normal crossing support, and meeting the strict transform of D also normally. We will see shortly that this context is well adapted for our purposes.

Let us recall that in the case where (X, D) is a log-smooth pair (with D having coefficients in $(0, 1)$) and $\pm(K_X + D)$ is ample, then the behavior of any Kähler-Einstein metric along D is well understood: it has so-called cone singularities (cf [**Bre11, CGP11, Don12, JMR11**]). We now want to show a similar statement in our situation; so let us first recall the notion of cone singularities.

3.3.2. Metrics with cone singularities. — Let X be a compact Kähler manifold of dimension n, and $D = \sum a_i D_i$ an effective $\mathbb{R}$-divisor with simple normal crossing support such that the a_i's satisfy the following inequality: $0 < a_i < 1$. We write $X_0 = X \setminus \text{Supp}(D)$, and we choose non-zero global sections s_i of $\mathcal{O}_X(D_i)$.

Our local model is given by the product $X_{\rm mod} = (\mathbb{D}^*) \times \mathbb{D}^{n-r}$ where $\mathbb{D}$ (resp. $\mathbb{D}^*$) is the disc (resp. punctured disc) of radius $1/2$ in $\mathbb{C}$, the divisor being $D_{\rm mod} = d_1[z_1 = 0] + \cdots + d_r[z_r = 0]$, with $d_i < 1$. We will say that a metric ω on $X_{\rm mod}$ has cone singularities along the divisor $D_{\rm mod}$ if there exists $C > 0$ such that

$$C^{-1}\omega_{\rm mod} \leqslant \omega \leqslant C\,\omega_{\rm mod}$$

where

$$\omega_{\rm mod} := \sum_{j=1}^{r} \frac{idz_j \wedge d\bar{z}_j}{|z_j|^{2d_j}} + \sum_{j=r+1}^{n} idz_j \wedge d\bar{z}_j$$

is simply the product metric of the standard cone metric on $(\mathbb{D}^*)^r$ and the euclidian metric on $\mathbb{D}^{n-r}$.

This notion makes sense for global (Kähler) metrics ω on the manifold X_0; indeed, we can require that on each trivializing chart of X where the pair (X, D) becomes $(X_{\rm mod}, D_{\rm mod})$ (those charts cover X), ω is equivalent to $\omega_{\rm mod}$ just like above; of course this does not depend on the chosen chart.

In our case, we are going to deal with Kähler metrics no more on the whole X_0 but on some Zariski open subset, more precisely $X_0 \cap \mathrm{Amp}(\alpha)$ for some semi-positive and big (or nef and big) class α (in Theorem B, we work on $X_0 \setminus \mathrm{Supp}(E)$ for example). Therefore one needs to make precise what we will call cone singularities for such a metric. Indeed, $\mathrm{Amp}(\alpha)$ being non compact in general (more precisely as soon as α is not ample), the bi-Lipschitz constant comparing the initial and the model cone metric in each local charts may not be chosen uniformly for all charts covering $\mathrm{Amp}(\alpha)$. So we do not require any kind of uniformity for these constants, and we will only focus on what happens on compact subsets of $X_0 \cap \mathrm{Amp}(\alpha)$.

3.3.3. Statement of the main result. — We have seen in the introduction that Theorem A – which asserts that (under some assumptions) the Kähler-Einstein metric for a klt pair has cone singularities along the boundary (when restricted to some suitable open subset) – is related to some particular properties of solutions of degenerate Monge-Ampère equations. So let us now fix the set-up.

Let X be a compact Kähler manifold of dimension n, α a nef and big class class and $\theta \in \alpha$ a smooth representative. Let $E = \sum c_j E_j$ be an effective $\mathbb{R}$-divisor with snc support such that $\alpha - E$ is Kähler. Let also $D = \sum a_i D_i$ an effective divisor with snc support such that E and D have no common components, and that $E + D$ has snc support.

We write $X_0 = X \setminus (\mathrm{Supp}(E) \cup \mathrm{Supp}(D))$, we choose non-zero global sections t_j of $\mathcal{O}_X(E_j)$ and s_i of $\mathcal{O}_X(D_i)$, and we choose some real numbers $b_j > -1$. We choose some smooth hermitian metrics on those bundles which we normalize so that $\int_X \prod |t_j|^{2b_j} \prod |s_i|^{-2a_i} dV = \mathrm{vol}(\alpha)$. Finally, if φ is a θ-psh function, we denote by $\mathrm{MA}(\varphi)$ the non-pluripolar product $\langle(\theta + dd^c\varphi)^n\rangle$.

Theorem 3.3.1. — *We assume that the coefficients of D satisfy the inequalities $a_i \geqslant 1/2$ for all i. Then any solution with full Monge-Ampère mass of*

$$\mathrm{MA}(\varphi) = \prod |t_j|^{2b_j} \frac{e^{\lambda\varphi} dV}{\prod |s_i|^{2a_i}}$$

defines a smooth metric on X_0 having cone singularities along D.

3.4. Proof of Theorem B

The strategy of the proof is to regularize the Monge-Ampère equation into an equation with smooth right-hand side. The stake will then consist in obtaining uniform estimates on the compact subsets of X_0 (cf Proposition (3.4.1)). In order to simplify the notations, we will use the same notation for $\mathrm{Supp}(E)$ and E as long as no confusion results from this.

3.4.1. Regularization. — We start from our solution φ, and we regularize the Monge-Ampère equation in two ways: first, we regularize φ by approximating it with a decreasing sequence of smooth quasi-psh functions τ_ε satisfying

$$dd^c \tau_\varepsilon \geqslant -C\omega \tag{3.4.1}$$

for some (Kähler) form ω. This is possible thanks to Demailly's regularization theorem [**Dem82, Dem92**]. In particular the τ_ε's are uniformly upper bounded by $\sup \tau_1$ for instance. Then, we consider the following equation (in φ_ε):

$$\langle (\theta + dd^c \varphi_\varepsilon)^n \rangle = \prod |t_j|^{2b_j} \frac{e^{\lambda \tau_\varepsilon} dV}{\prod (|s_i|^2 + \varepsilon^2)^{a_i}} \tag{MA_ε}$$

By multiplying dV with a constant, we can make sure that the total mass of the RHS is $\mathrm{vol}(\alpha)$); this constant depends on ε, but in a totally harmless way because $\frac{e^{\lambda\tau_\varepsilon} \prod |t_j|^{2b_j}}{\prod (|s_i|^2+\varepsilon^2)^{a_i}}$ is uniformly bounded in $L^1(dV)$: this is clear if $\lambda \geqslant 0$ and follows from the monotone convergence combined with Theorem 3.2.1 if $\lambda < 0$: indeed, this result of [**BBE**$^+$**11**] shows that $e^{-\varphi} \in L^p(dV)$ for all $p \geqslant 1$. Therefore we can assume that the volume form is already normalized.

By [**BEGZ10**], we know that (MA_ε) has a unique solution φ_ε which is θ-psh and has minimal singularities. One could also deduce a uniform estimate, but at that point we do not really need it, and we will anyway recover it implicitly with Proposition 3.4.1.

3.4.2. Laplacian estimates. — In this section, we explain in Proposition 3.4.1 how to obtain laplacian estimates for our regularized solutions: this is the key result for the proof of the main theorems. It is an adaptation of [**BBE**$^+$**11**, Theorem 10.1] in the nef and big case, with a slight refinement (cf point (iii)) which will be crucial for us. Before we can state the result, let us introduce some notation.

We recall that α is a nef and big class and E is an effective $\mathbb{R}$-divisor such that $\alpha - E$ is Kähler. We choose s_E a non-zero section of the $\mathcal{O}_X(E)$, and choose some smooth hermitian metric h on this $\mathbb{R}$-line bundle; we set

$$\rho := \log |s_E|_h^2$$

(actually s_E is not well-defined if E is not a $\mathbb{Z}$-divisor, but ρ is, by $\mathbb{R}$-linearity). Then, we define

$$\omega_\rho := \theta - \Theta_h(E)$$

If h is properly chosen, then ω_ρ is a Kähler form ($\Theta_h(E)$ denotes the Chern curvature of the hermitian $\mathbb{R}$-line bundle $(\mathcal{O}_X(E), h)$, which is also defined by $\mathbb{R}$-linearity). As

$$\theta + dd^c\rho = \omega_\rho + [E]$$

we see that ω_ρ coincides with $\theta + dd^c\rho$ on $X \setminus E$.

We emphasize that ω_ρ depends on the smooth hermitian metric chosen on E, or equivalently it depends on ρ, hence the notation. In the following proposition, we make explicit the precise dependence in h (or ρ) in the laplacian estimates obtained in [**BBE$^+$11**, Theorem 10.1]:

Proposition 3.4.1. — *With the previous notations, let $\psi^\pm$ be quasi-psh functions on X such that $e^{-\psi^-} \in L^p(dV)$ for some $p > 1$, and satisfying $\int_X e^{\psi^+ - \psi^-}\omega_\rho^n = \mathrm{vol}(\alpha)$. Let φ be a θ-psh function solution of*

$$\langle(\theta + dd^c\varphi)^n\rangle = e^{\psi^+ - \psi^-}\omega_\rho^n$$

and assume given a constant $C > 0$ such that

(i) $dd^c\psi^+ \geqslant -C\omega_\rho$ *and* $\sup_X \psi^+ \leqslant C$;
(ii) $dd^c\psi^- \geqslant -C\omega_\rho$ *and* $\|e^{-\psi^-}\omega_\rho^n/dV\|_{L^p(dV)} \leqslant C$;
(iii) $\omega_\rho \geqslant C^{-1}\omega$ *and the holomorphic bisectional curvature of* ω_ρ *is bounded below on* X *by* $-C$.

Then there exists $A, B > 0$ depending only on θ, p and C such that on $X \setminus E$:

$$0 \leqslant \theta + dd^c\varphi \leqslant Ae^{-B\rho - \psi^-}\omega_\rho$$

Proof. — We will only treat the case where $\psi^\pm$ are smooth; the general case can be reduced to the smooth case by regularization exactly in the same way as in [**BBE$^+$11**], the only difference being the use of the degenerate version of Kołodziej's stability theorem for *big* classes given in [**GZ11**, Theorem C].

For $t > 0$ (say $t \leqslant 1$), we consider the Kähler form ω_t on X defined by $\omega_t := (1 + t)\omega_\rho$; we also define $\theta_t := \theta + t\omega_\rho$. As α is nef, $\{\theta_t\}$ is a Kähler class for all $t > 0$. Therefore there exists a unique normalized (smooth) θ_t-psh function φ_t such that

$$(\theta_t + dd^c\varphi_t)^n = e^{\psi^+ - \psi^-}e^{c_t}\omega_\rho^n$$

where $e^{c_t} = \mathrm{vol}(\alpha + t\{\omega_\rho\})/\mathrm{vol}(\alpha)$. As $\{\omega_\rho\}$ is independent of ρ (E is fixed), c_t is uniformly bounded.

We want to obtain an estimate $|\Delta_{\omega_\rho}\varphi_t| \leqslant Ae^{-B\rho - \psi^-}$ on $X \setminus E$; as $\omega_\rho \geqslant \frac{1}{2}\omega_t$, it will be enough to show the same estimate with $\Delta_{\omega_t}\varphi_t$.

To begin with, thanks to assumptions (i) and (ii), we have a $\mathscr{C}^0$ estimate given by [**BEGZ10**], and recalled in section 3.2.2.2: $\varphi_t \geqslant V_{\theta_t} - M$. Besides, it is easy to see that V_{θ_t}

decreases to V_θ when $t \to 0$, so that $\varphi_t \geqslant V_\theta - M$ for all $t \geqslant 0$.

For the rest of the proof, we will work on $X \setminus E$, so that ω_ρ actually coincides with $\theta + dd^c\rho$.

At this point of the proof, one cannot use Siu's formula [**Siu87**, pp. 98-99] as in [**BBE$^+$11**] anymore because we do not have a uniform upper bound on the scalar curvature of ω_t. Instead, we use a variant of Siu's formula given in [**CGP11**, Lemma 2.2] involving only a lower bound on the bisectional curvature of ω_t; it yields:

$$\Delta_{\omega'_t}\left(\log \mathrm{tr}_{\omega_t}(\omega'_t)\right) \geqslant \frac{\Delta_{\omega_t}(\psi^+ - \psi^-)}{\mathrm{tr}_{\omega_t}(\omega'_t)} - C\mathrm{tr}_{\omega'_t}(\omega_t)$$

where $\omega'_t = \theta_t + dd^c\varphi_t$. We notice, as in [**BBE$^+$11**], that even if [**CGP11**, Lemma 2.2] is stated for two cohomologous forms, the last formula is valid because the computations are local, and locally all forms are cohomologous.

Using both inequalities $\omega_t \geqslant \omega_\rho$ and $n \leqslant \mathrm{tr}_{\omega_t}(\omega'_t)\mathrm{tr}_{\omega'_t}(\omega_t)$, we get then a constant $A_0 > 0$ under control such that:

$$\Delta_{\omega'_t}\left(\log \mathrm{tr}_{\omega_t}(\omega'_t)\right) \geqslant -\frac{\Delta_{\omega_t}\psi^-}{\mathrm{tr}_{\omega_t}(\omega'_t)} - A_0\mathrm{tr}_{\omega'_t}(\omega_t)$$

Now we use the computations of [**BBE$^+$11**] based on [**Pău08**, Lemma 3.2]: as $C\omega_t + dd^c\psi^- \geqslant 0$, we have

$$0 \leqslant C\omega_t + dd^c\psi^- \leqslant \mathrm{tr}_{\omega'_t}(C\omega_t + dd^c\psi^-)\omega'_t$$

Taking the trace with respect to ω_t gives:

$$0 \leqslant nC + \Delta_{\omega_t}\psi^- \leqslant \left(C\mathrm{tr}_{\omega'_t}(\omega_t) + \Delta_{\omega'_t}\psi^-\right)\mathrm{tr}_{\omega_t}(\omega'_t)$$

so that

$$\Delta_{\omega'_t}\psi^- \geqslant -\frac{nC + \Delta_{\omega_t}\psi^-}{\mathrm{tr}_{\omega_t}(\omega'_t)} - C\mathrm{tr}_{\omega'_t}(\omega_t)$$

and therefore:

$$\Delta_{\omega'_t}\left(\log \mathrm{tr}_{\omega_t}(\omega'_t) + \psi^-\right) \geqslant -A_1\mathrm{tr}_{\omega'_t}(\omega_t) \tag{3.4.2}$$

for some constant $A_1 > 0$ under control.

Now, if we set

$$u_t := \varphi_t - \rho$$

then $\omega'_t = \omega_t + dd^c u_t$ so that $n = \mathrm{tr}_{\omega'_t}(\omega_t) + \Delta_{\omega'_t}u_t$. Equation (3.4.2) gives us two positive constants A_2, A_3 under control satisfying:

$$\Delta_{\omega'_t}(\log \mathrm{tr}_{\omega_t}(\omega'_t) + \psi^- - A_2u_t) \geqslant \mathrm{tr}_{\omega'_t}(\omega_t) - A_3$$

We want now to apply as usual the maximum principle to the term inside the laplacian in the right hand side. To ensure we can do this, we must check that the function

$$H_t := \log \mathrm{tr}_{\omega_t}(\omega'_t) + \psi^- - A_2u_t$$

attains its maximum on $X \setminus E$. This is a qualitative problem, and of course we do not ask any kind of uniformity here. We know that ψ^- is bounded (but we do not have uniform bounds), moreover $\mathrm{tr}_{\omega_t}(\omega'_t) \leqslant C\mathrm{tr}_\omega(\omega'_t)$ by assumption (iii), and as ω'_t is smooth, this last quantity is bounded above. Finally, $-u_t = \rho - \varphi_t$ is upper bounded and tends to $-\infty$ near E. We conclude that H_t attains its maximum on at some point $x_t \in X \setminus E$, so that $\mathrm{tr}_{\omega'_t}(\omega_t)(x_t) \leqslant A_3$ is under control.

Using the basic inequality

$$\mathrm{tr}_{\omega_t}(\omega'_t) \leqslant n\,(\omega'^n_t/\omega^n_t)\left(\mathrm{tr}_{\omega'_t}(\omega_t)\right)^{n-1}$$

and the inequality

$$\omega'^n_t/\omega^n_t = e^{\psi^+-\psi^-}\frac{e^{c_t}\omega^n_\rho}{\omega^n_t} \leqslant e^{c_t+\psi^+-\psi^-}$$

we get

$$\log\mathrm{tr}_{\omega_t}(\omega'_t) \leqslant -\psi^- + (n-1)\log\mathrm{tr}_{\omega'_t}(\omega_t) + A_4$$

with A_4 under control, and therefore

$$H_t \leqslant (n-1)\log\mathrm{tr}_{\omega'_t}(\omega_t) - A_2u_t + A_4$$

so that

$$\sup_{X\setminus E} H = H(x_t) \leqslant A_5 - A_2u_t(x_t)$$

Therefore, one has, for any $x \in X \setminus E$:

$$\begin{aligned}(\log\mathrm{tr}_{\omega_t}(\omega'_t) + \psi^-)(x) &= H(x) + A_2u_t(x)\\ &\leqslant H(x_t) + A_2u_t(x)\\ &\leqslant A_5 + A_2(u_t(x) - u_t(x_t))\\ &\leqslant A_6 + A_2u_t(x)\end{aligned}$$

Indeed, $\varphi_t \geqslant V_\theta - M$ and ρ is θ-psh, thus $\varphi_t \geqslant \rho - A_7$ so that $\inf_{X\setminus E}\varphi_t - \rho$ is uniformly bounded from below. As φ_t is normalized, $u_t \leqslant -\rho$, so that one finally gets $A, B > 0$ under control and satisfying: $\log\mathrm{tr}_{\omega_t}(\omega'_t) + \psi^- \leqslant A - B\rho$, which is what we were looking for. □

Remark 3.4.2. — Combined with Evans-Krylov's theorem, this proposition shows that any Kähler-Einstein metric attached to a klt pair (X, D) (satisfying e.g. $K_X + D$ ample, the other cases being similar) is smooth on $X_{\mathrm{reg}} \setminus \mathrm{Supp}(D)$. Indeed, if we work on a suitable log-resolution $X' \xrightarrow{\pi} X$ of (X, D) with $K_{X'} + D' = \pi^*(K_X + D)$, then the KE metric (viewed on X') written as usual $\theta' + dd^c\varphi'$ satisfies an equation of the form $(\theta' + dd^c\varphi')^n = e^{\varphi'-\varphi'_D}dV)$ and we just have to apply the previous proposition with $(\psi^+, \psi^-) = (\varphi' + \varphi_{D'_-}, \varphi_{D'_+})$ if $D' = D'_+ - D'_-$ is the decomposition of D' into its positive and negative part.

3.4.3. Approximation of the cone metric. — We now recall the global approximation of a cone metric, as explained in [**CGP11**, Section 3] for instance.
Let ω be a Kähler form on some compact Kähler manifold Y carrying a $\mathbb{R}$-divisor $F = \sum c_kF_k$ with simple normal crossing support and coefficients $c_k \in]0, 1[$. Then for any sufficiently small $\varepsilon > 0$, there exists a smooth function ψ_ε such that the form ω_ε on Y defined by

$$\omega_\varepsilon := \omega + dd^c\psi_\varepsilon$$

satisfies the following properties:

· ω_ε dominates a fixed Kähler form on Y;
· ψ_ε is uniformly bounded (on Y) in ε;
· When ε goes to 0, ω_ε converges to some Kähler metric on $Y \setminus \mathrm{Supp}(F)$ having cone singularities along F.

Our goal is now to apply this construction for some suitable Kähler form ω, and try to apply the laplacian estimates obtained in the previous section in order to get $C^{-1}\omega_\varepsilon \leqslant \theta + dd^c\varphi_\varepsilon \leqslant C\omega_\varepsilon$ on compact subsets of $X \setminus E$.

We know that for some hermitian metric h on E, the form $\omega_\rho = \theta - \Theta_h(E)$ is a Kähler form. So we can apply the previous approximation process to $Y = X$, $F = D$, and $\omega = \omega_\rho$. We get a sequence of smooth functions ψ_ε such that $\omega_{\rho_\varepsilon} := \omega_\rho + dd^c\psi_\varepsilon$ is a Kähler form satisfying the three conditions above. Moreover, $\omega_{\rho_\varepsilon}$ corresponds to the Kähler metric $\theta - \Theta_{h_\varepsilon}(E)$ where $h_\varepsilon := he^{\psi_\varepsilon}$, or equivalently $\rho_\varepsilon := \rho + \psi_\varepsilon$, and $\omega_{\rho_\varepsilon}$ still coincides with $\theta + dd^c\rho_\varepsilon$ on $X \setminus E$.

3.4.4. End of the proof. — Recall now that we try to understand the behaviour of the solution of

$$(\mathrm{MA}_\varepsilon) \qquad \langle(\theta + dd^c\varphi_\varepsilon)^n\rangle = \prod |t_j|^{2b_j} \frac{e^{\lambda\tau_\varepsilon} dV}{\prod(|s_i|^2 + \varepsilon^2)^{a_i}}$$

If we use the metric $\omega_{\rho_\varepsilon}$ as new reference, equation $(\mathrm{MA}_\varepsilon)$ may be rewritten:

$$\langle(\theta + dd^c\varphi_\varepsilon)^n\rangle = \prod |t_j|^{2b_j} e^{\lambda\tau_\varepsilon + F_\varepsilon} \omega_{\rho_\varepsilon}^n$$

where

$$F_\varepsilon = \log\left(\frac{dV}{\prod(|s_i|^2 + \varepsilon^2)^{a_i} \omega_{\rho_\varepsilon}^n}\right).$$

In order to use the same notations as Proposition 3.4.1, we set:

$$\psi^+ := \sum_{b_j > 0} b_j \log|t_j|^2 + \lambda\tau_\varepsilon + F_\varepsilon, \quad \psi^- := \sum_{b_j < 0} -b_j \log|t_j|^2$$

if $\lambda \geqslant 0$, and

$$\psi^+ := \sum_{b_j > 0} b_j \log|t_j|^2 + F_\varepsilon, \quad \psi^- := -\lambda\tau_\varepsilon + \sum_{b_j < 0} -b_j \log|t_j|^2$$

if $\lambda < 0$. We should add that despite the notations, ψ^+ and ψ^- actually depend on ε. With these notations, equation $(\mathrm{MA}_\varepsilon)$ becomes:

$$(\mathrm{MA}'_\varepsilon) \qquad \langle(\theta + dd^c\varphi_\varepsilon)^n\rangle = e^{\psi^+ - \psi^-} \omega_{\rho_\varepsilon}^n$$

On the compact subsets of $X \setminus E$, $\omega_{\rho_\varepsilon}$ converge to a Kähler metric with cone singularities along D. Therefore the proof of Theorem B boils down to showing that on each relatively compact open subset $U \Subset X \setminus E$, there exists a constant $C_U > 0$ such that on U, we have for each $\varepsilon > 0$:

$$C_U^{-1}\omega_{\rho_\varepsilon} \leqslant \theta + dd^c\varphi_\varepsilon \leqslant C_U\omega_{\rho_\varepsilon}$$

We fix such a open subset U. We want to apply Proposition 3.4.1 in our situation, so we need to check that there exists $C > 0$ independent of ε satisfying:

(i) $dd^c\psi^+ \geqslant -C\omega_{\rho_\varepsilon}$ and $\sup_X \psi^+ \leqslant C$;

(ii) $dd^c\psi^- \geqslant -C\omega_{\rho_\varepsilon}$ and $\|e^{-\psi^-}\omega_{\rho_\varepsilon}^n/dV'\|_{L^p} \leqslant C$;

(iii) $\omega_{\rho_\varepsilon} \geqslant C^{-1}\omega$ and the holomorphic bisectional curvature of $\omega_{\rho_\varepsilon}$ is bounded from below (on X) by $-C$.

Let us begin with (ii). The first thing to check is that $dd^c\tau_\varepsilon \geqslant -C\omega_{\rho_\varepsilon}$. As $\omega_{\rho_\varepsilon}$ dominates some uniform Kähler form (cf construction of the approximate cone metric, section 3.4.3), this follows from (3.4.1). Then, we have to check the uniform L^p integrability condition. We may assume that $\lambda \leqslant 0$; let us fix $\delta > 0$ small enough, and more precisely

$$\delta < \min_j \frac{1}{b_j^-} - 1$$

where $b_j^- = \max(-b_j, 0)$.
Then by monotone convergence, $e^{\lambda(1+\delta)\tau_\varepsilon}$ is uniformly in L^q for all $q \geqslant 1$ because so is $e^{\lambda(1+\delta)\varphi}$ by Theorem 3.2.1; we will fix an appropriate q later. Moreover, by the normal crossing property and the klt condition, $(\prod_{b_j<0}|t_j|^{2b_j})^{1+\delta}$ is also uniformly in $L^{1+\eta}(dV)$ for some $\eta > 0$ small enough. Therefore, taking $q = 1 + \frac{1}{\eta}$, we get the result.

There remain two non-trivial estimates to check, namely that $dd^cF_\varepsilon \geqslant -C\omega_{\rho_\varepsilon}$, and that the holomorphic bisectional curvature of $\omega_{\rho_\varepsilon}$ is uniformly bounded below on $X \setminus E$. This is precisely at this point of the proof that we use in a crucial manner the assumption that the coefficients of D are in $[\frac{1}{2}, 1)$.

Indeed the first estimate is already obtained in [**CGP11**, §4.5] in the slightly weaker form $\Delta_{\omega_{\rho_\varepsilon}} F_\varepsilon \geqslant -C$, and is proven in the desired form in [**Gue12b**, §4.2.3]. As for the second estimate, concerning the curvature of $\omega_{\rho_\varepsilon}$, it is also proven in [**CGP11**, §4.3-4.4]. Finally, as for the bound $F_\varepsilon \leqslant C$, it is quite easy and explained in the same references.

Therefore, we can legitimately apply Proposition 3.4.1 to equation $(\mathrm{MA}'_\varepsilon)$, which ends the proof of Theorem B.

Remark 3.4.3. — Let us emphasize that we may use the computations of [**CGP11**] to estimate e.g. the curvature of $\omega_{\rho_\varepsilon}$ because this last metric is of the form $\omega_\rho + dd^c\psi_\varepsilon$ for some *fixed* Kähler metric ω_ρ on the whole X. It is not clear to us whether the same regularization argument could be performed directly on $\mathrm{Amp}(\alpha)$ (ie without choosing a suitable compactification) as in [**BBE$^+$11**, Theorem 10.1].

3.4.5. A slight generalization. — In the course of the proof of Theorem B, we do not really use the fact that the factors $|t_j|^{-2b_j}$ in the RHS of the Monge-Ampère equation correspond to divisors $(t_j = 0)$ included in the "non-ample part" of α, denoted by E. Therefore, one could equally choose t_j to be a section of $\mathcal{O}_X(D_j)$ for some component D_j of D with coefficient $a_j < 1/2$. This leads to the following generalization of Theorem A:

Theorem 3.4.4. — *Let (X, D) be a klt pair. We write $D = D_{\geqslant \frac{1}{2}} + D_{<\frac{1}{2}} = \sum_{a_i \geqslant \frac{1}{2}} D_i + \sum_{a_i < \frac{1}{2}} D_i$, and we set* $\mathrm{LS}(X, D_{\geqslant\frac{1}{2}}) := \{x \in X; (X, D_{\geqslant\frac{1}{2}})$ *is log-smooth at* $x\}$.

(*i*) *If* $K_X + D$ *is big, then the Kähler-Einstein metric of* (X, D) *has cone singularities along* $D_{\geqslant \frac{1}{2}}$ *on* $\mathrm{LS}(X, D_{\geqslant \frac{1}{2}}) \cap \mathrm{Amp}(K_X + D) \setminus \mathrm{Supp}(D_{<\frac{1}{2}})$.

(*ii*) *If* $-(K_X + D)$ *is ample, then any Kähler-Einstein metric of* (X, D) *has cone singularities along* $D_{\geqslant \frac{1}{2}}$ *on* $\mathrm{LS}(X, D_{\geqslant \frac{1}{2}}) \setminus \mathrm{Supp}(D_{<\frac{1}{2}})$.

CHAPITRE 4

KÄHLER-EINSTEIN METRICS ON STABLE VARIETIES AND LOG CANONICAL PAIRS

Le chapitre qui suit est issu de l'article [**BG13**] écrit en collaboration avec Robert Berman.

Pour faire le lien avec le chapitre précédent, nous continuons notre analyse des métriques de Kähler-Einstein sur des variétés singulières, mais cette fois sous une autre perspective. En effet, au chapitre 3, le cadre était celui des paires klt pour lesquelles l'existence de métriques KE était déjà connue, et nous nous intéressions alors à leur régularité le long du diviseur de bord.

Maintenant, nous allons nous intéresser aux paires (semi-)log canoniques pour lesquelles aucun résultat d'existence n'était connu auparavant, éventuellement mis à part le cas log lisse. Malgré leurs définitions très proches, le cadre log canonique (lc) est très différent du cadre klt, aussi bien du point de vue algébrique qu'analytique. En particulier, les méthodes existant jusqu'alors ne permettent pas de traiter le problème de l'existence de métriques de Kähler-Einstein, notamment à cause du caractère non borné du potentiel de telles métriques.

En entremêlant les méthodes variationnelles et les méthodes classiques d'estimées a priori, nous allons montrer qu'une paire (semi-)log canonique (X, D) telle que $K_X + D$ est ample admet une unique métrique de Kähler-Einstein à courbure négative. De plus, toujours sous cette condition de positivité, nous montrerons que l'existence d'une métrique KE sur une variété (semi-)normale est équivalente à ce que ses singularités soient (semi-)log canoniques.

Introduction

According to the seminal works of Aubin [**Aub78**] and Yau [**Yau78b**] any *canonically polarized* compact complex manifold X (i.e. X is a non-singular projective algebraic variety such that the canonical line bundle K_X is ample) admits a unique Kähler-Einstein metric ω in the first Chern class $c_1(K_X)$. One of the main goals of the present paper is to extend this result to the case when X is singular or more precisely when X has *semi-log canonical* singularities. A major motivation comes from the fact that such singular varieties appear naturally in the compactification of the moduli space of canonically polarized manifolds - a subject where there has been great progress in the last years in connection to the (log) Minimal Model Program (MMP) in birational algebraic geometry [**Kol, Kov12**]. The varieties in question are usually refered to as *stable varieties* (or *canonical models*) as they are the higher dimensional generalization of the classical notion of stable curves of genus $g > 1$, which form the Deligne-Mumford compactification of the moduli space of non-singular genus g curves [**Kol, Kov12**]. It as a classical fact that any stable curve admits a unique Kähler-Einstein metric on its regular part, whose total area is equal to the (arithmetic) degree of the curve X and our first main result gives a generalization of this fact to the higher dimensional setting:

Theorem A. — *Let X be a projective complex algebraic variety with semi-log canonical singularities such that K_X is ample. Then there exists a Kähler metric on the regular locus X_{reg}, satisfying*

$$\operatorname{Ric}\omega = -\omega$$

and such that the volume of $(X_{\mathrm{reg}}, \omega)$ coincides with the volume of K_X, i.e. $\int_{X_{\mathrm{reg}}} \omega^n = c_1(K_X)^n$. Moreover, the metric extends to define a current ω in $c_1(K_X)$ which is uniquely determined by X.

We will refer to the current ω in the previous theorem as a (singular) Kähler-Einstein metric on X. Moreover, the current ω will be shown to be of *finite energy,* in the sense of [**GZ07, BBGZ09**] and as discussd in the last section of the present paper this allows one to define a canonical (singular) Weil-Peterson metric on the compact moduli space in terms of Deligne pairings. The notion of semi-log canonical singularities of a variety X - which is the most general class of singularities appearing in the (log) Minimal Model Program - will be recalled below. For the moment let us just point out that the definition involves two ingredients: first a condition which makes sure that the canonical divisor K_X is defined as a $\mathbb{Q}-$ Cartier divisor (i.e. $\mathbb{Q}-$line bundle) which is in particular needed to make so sense of the notion of ampleness of K_X and secondly, the definition of semi-log canonical singularities involves a bound on the discrepancies of X on any resolution of singularities.

In fact, we will conversely show that if K_X is ample and the variety X admits a Kähler-Einstein metric then X has semi-log canonical singularities and this brings us to our second motivation for studying Kähler-Einstein metrics in the the singular setting, namely the Yau-Tian-Donaldson conjecture. Recall that this conjecture concerns polarized algebraic manifolds (X, L), i.e. algebraic manifolds together with an ample line bundle $L \to X$ and it says that the first Chern class $c_1(L)$ of L contains a Kähler metric ω with constant scalar curvature if and only if (X, L) is K-stable. The latter notion of stability is of an algebro-geometric nature and can be seen as an asymptotic form of the classical notions of Chow and Hilbert stability appearing in Geometric Invariant Theory (GIT). However, while the notion of K-stability makes equal sense when X is singular it is less clear how to give a proper definition of a constant scalar curvature metric for a singular polarized variety (X, L). But, as it turns out, the situation becomes more transparent in the case when L is equal to K_X or its dual, the anti-canonical bundle $-K_X$. The starting point is the basic fact that, when X is smooth, a Kähler metric in ω in $c_1(\pm K_X)$ has constant scalar curvature on all of X precisely when it has constant Ricci curvature, i.e. when ω is a Kähler-Einstein metric. Various generalizations of Kähler-Einstein metrics to the singular setting have been proposed in the litterature, see e.g. [**EGZ09, BEGZ10, BBE$^+$11**] etc. In this paper we will adopt the definition which appears in the formulation of the previous theorem (see section 4.1), i.e. positive current in $c_1(\pm K_X)$ is said to define a (singular) Kähler-Einstein metric if defines a bona fide Kähler-Einstein metric on the regular locus X_{reg} and if its total volume there coincides with the algebraic top intersection number of $c_1(\pm K_X)$. This definition, first used in the Fano case in [**BBE$^+$11**], has the virtue of generalizing all previously proposed definitions, regardless of the sign of the canonical line bundle. Combing our results with recent results of Odaka [**Oda08, Oda11**], which say that a canonicaly polarized variety has semi-log canonical singularities precisely when (X, K_X) is K-stable, gives the following theorem, which can be seen as a confirmation of the generalized form of the Yau-Tian-Donaldson conjecture for canonically polarized varieties (satisfying the conditions G_1 and S_2, cf 4.2.10):

Theorem B. — *Let X be a projective complex algebraic variety such that K_X is ample. Then X admits a Kähler-Einstein metric if and only if (X, K_X) is $K-$stable.*

It may also be illuminating to compare this result with the case when $L := -K_X$ is ample (i.e. X is Fano). Then it was shown in [**Ber12**], in the general singular setting, that the existence of a Kähler-Einstein metrics indeed implies K-(poly)stability. As for the converse it was finally settled very recently in the deep works by Chen-Donaldson-Sun [**CDS12a, CDS12b, CDS13**] and Tian [**Tia13**], independently, in the case when X is smooth. The existence problem in the singular case is still open in general, except for the toric case [**BB12**]; cf also [**OSS12**] for a related problem in the case of singular Fano surfaces.

Coming back to the present setting we point out that the starting point of our approach is that, after passing to a suitable resolution of singularities, we may as well assume that the variety X is smooth if we work in the setting of *log pairs* (X, D), where D is a $\mathbb{Q}-$divisor on X with simple normal crossings (SNC) and where the role of the canonical line bundle is played by the *log canonical line bundle* $K_X + D$ (which appears as the pull-back to the resolution of the original canonical line bundle). In this notation the original variety has semi-log canonical singularities precisely when the log pair (X, D) is *log canonical (lc)* in the usual sense of the Minimal Model Program, i.e. the coefficents of D are at most equal to one (but negative coefficents are allowed). However, it should be stressed that for this gain in regularity we have, of course, to pay a loss of positivity: even if the original canonical line bundle is ample, the corresponding log canonical line bundle is only *semi-ample* (and big) on the resolution, since it is trivial along the exceptional divisors of the corresponding resolution.

The upshot is that the natural setting for our results is the setting of log canonical pairs (X, D) such that the log canonical line bundle $K_X + D$ is semi-ample and big. To any such pair we will associate a canonical Khler-Einstein metric ω in the sense that ω is a current in the first Chern class $c_1(K_X + D)$ such that ω restricts to a bona fide Kähler-Einstein metric on a Zariski open set of X and such that, globally on X, the current defined by the divisor D gives a singular contribution to the Ricci curvature of ω.

The existence proof of Theorem A (and its generalizations described below) will be divided into two parts: in the first part we construct a *variational solution* with *finite energy*, by adapting the variational techniques developed in [**BBGZ09**] to the present setting. Then, in the second part, we show that the variational solutions have appopriate regularity using a priori Laplacian estimates, building on the works of Aubin [**Aub78**] and Yau [**Yau78b**] and ramifications of their work by Kobayashi [**Kob84**] and Tian-Yau [**TY87**] to the setting of quasi-projective varieties - in particular we will be relying on Yau's maximum principle. For the second part we will need to perturb the line bundle $L := K_X + D$ (to make it ample) and regularize the klt part of the divisor D (to make the divisor purely log canonical). To this end it will be convenient to consider the even more general setting of *twisted Kähler-Einstein metrics* attached to a *twisted log canonical pairs* (X, D, θ) where (X, D) is a log canonical pair and θ is a smooth form on X such that the twisted log canonical line bundle $K_X + D + [\theta]$ is semi-positive and big (in the perturbation argument it will be ample/Kähler). Then the twisted Kähler-Einstein metric is defined as before, after replacing $\mathrm{Ric}\,\omega$ with the twisted Ricci curvature $\mathrm{Ric}\,\omega - \theta$.

Theorem C. — *Let X be a Kähler manifold and D a simple normal crossings $\mathbb{R}-$divisor on X with coefficients in $]-\infty, 1]$ such that $K_X + D$ is semi-positive and big (i.e. $(K_X + D)^n > 0$). Then there exists a unique current ω in $c_1(K_X + D)$ which is smooth on a Zariski open set U of X and such that*

$$\mathrm{Ric}\,\omega = -\omega + [D]$$

holds on X in the weak sense and $\int_U \omega^n = (K_X + D)^n$. More precisely, U can be taken to be the complement of D in the ample locus of $K_X + D$. Moreover,

- *Any such current ω on X automatically has finite energy.*
- *ω can be approximated by complete approximate Kähler-Einstein metrics ω_j in the following sense: split $D = D_{lc} + D_{klt}$ where D_{lc} is the the purely log canonical part and let θ_j be a sequence of smooth forms regularizing the current D_{klt} and fix a Kähler form ω_0 on X. Then the twisted Kähler-Einstein metrics ω_j attached to $(X, D_{lc}, \theta_j + \frac{1}{j}\omega_0)$ are smooth and complete on $X - D_{lc}$ and $\omega_j \to \omega$ in the weak toplogy of currents on X and in the $\mathcal{C}^\infty_{loc}$- topology on the Zariski open set U above.*

In the last section of the paper some applications of Theorem A are given. First, we explain the link with Yau-Tian-Donaldson as we indicated above in Theorem B. Then, we give a short analytic proof of the fact that the automorphism group of a canonically polarized variety with log canonical singularities is finite (see [**BHPS12**] for algebro-geometric proofs). We also discuss the problem of deducing Miyaoka-Yau type inequalities from Theorem A.

Further comparison with previous results. — The previous theorem also extends some of the results of Wu in [**Wu08, Wu09**], concerning the setting of Kähler-Einstein metrics on quasi-projective projective varieties of the form $X_0 := X - D$, where X is smooth and D is reduced SNC divisor. We recall that the case when $K_X + D$ is ample was independently settled by Kobayashi [**Kob84**] and Tian-Yau [**TY87**]. The case when X is an orbifold and $K_X + D$ is semi-ample and big was considered by Tian-Yau in [**TY87**] and as later shown by Yau [**Yau93**] the corresponding Kähler-Einstein metric is then complete on X_0. (in the orbifold sense). However, in our general setting the metric will typically not be complete on the regular locus. This is only partly due to the klt singularities (which generalize orbifold singularities) - there is also a complication coming from the presence of negative coefficents on a resolution (see section 4.4.7).

To illustrate this we recall that a standard example of log canonical pairs (X, D) is given by the Borel-Baily compactification $X := X_0 - D$ of an arithmetic quotient, i.e. $X_0 = B/\Gamma$, where B is a bounded symmetric domain and Γ is discrete subgroup of the automorphism group of B. In this case any toroidal resolution X' has the property that the corresponding divisor D' on the resolution is reduced (and hence purely log canonical) if Γ is neat, i.e. if there are no fixed points. The corresponding Kähler-Einstein metric on X_0 is the complete one induced from the corresponding metric on B, constructed in [**CY80, MY83**]. When Γ has fixed points these give rise to an additional fractional klt part D'_{klt} in D' so that the corresponding Kähler-Einstein metric is only complete in the orbifold sense [**TY87**]. However, for general log canonical singularity (X, D) the klt part D'_{klt} of D' may not be fractional or more seriously: it may contain negative coefficients and the main novelty of the present paper is to show how to deal with this problem by combining a variational approach with a priori estimates.

It is also interesting to compare with the case when the pair (X, D) is log smooth with $K_X + D$ ample and with D effective and klt (i.e. with coefficients in $[0, 1[$), where very precise regularity results have been obtained recently. For example, in [**Bre11, CGP11, JMR11**] it is shown that the corresponding Kähler-Einstein metric ω has conical singularities along D (sometimes also called edge singularities in the litterature), thus confirming a previous conjecture of Tian – note that the results of [**CGP11**] holds for coefficients bigger than $1/2$ whereas in [**JMR11**] the divisor D is assumed to be smooth but with no assumptions on the coefficients, and in [**Bre11**] both conditions are required; moreover the general case is announced in [**MR12**]. As for the mixed case when the coefficient 1 is also allowed in D it was studied in [**Gue12b**], where it was shown that ω has mixed cone and Poincaré type singularities. A commun theme in these results is that singularities of the metric ω are encoded by a suitable local model (with cone or Poincaré type singularities) determined by D. However, the difficulty in the situation studied in the present paper is the presence of negative coefficients in D and the associated loss of positivity which appears when we pass to a log resolution of a singular variety X. It would be very interesting if one could associate local models to this situation as well, but this seems very challenging even in the case when X has klt singularities.

Organization of the paper. —

- §4.1: We introduce the preliminary material that we will need, concerning the pluripotential theoretic setting of singular metrics on line bundles over varieties which are not necessarily normal.
- §4.2: Here we give the definition of a Kähler-Einstein metric on a canonically polarized variety X and more generally on a log pair (X, D). As we explain a purely differential-geometric definition can be given which only involves the regular locus X_{reg} of X. But, at we show, the corresponding metric automatically extends in a unique manner to define a singular current on X (which will allow us to prove the uniqueness of the Kähler-Einstein metric, later on in section 4.3). We first treat the case when X has log canonical (and hence normal singularities) and then the general case of a variety X with semi-log singularities. Anyway, as we recall, the latter case reduces to the former (if one works in the setting of pairs) if one passes to the normalization.
- §4.3: We prove the uniqueness and existence of a weak Kähler-Einstein metric in the general setting of varieties of log general type. The existence is proved by adapting the variational approach to complex Monge-Ampère equations introduced in [**BBGZ09**] to

the present setting. This method produces a singular Kähler-Einstein metric with finite energy (the new feature here compared to [**BBGZ09**] is that the reference measure does not have an L^1 density). We also use the variational approach to establish a stability result for the solutions to the equations induced from an (ample) perturbation of the log canonical line bundle on a resolution.

- §4.4: Here we establish the smoothness of the Kähler-Einstein metric, produced by the variational approach, on the regular locus of the variety X (or more generally, the pair (X, D)). The proof uses a perturbation argument in order to reduce the problem the the original setting of Kobayashi and Tian-Yau, combined with *a priori* estimates. But it should be stressed that in order to control the $\mathscr{C}^0$ norms we need to invoke the variational stability result proved in the previous section. We also investigate the (non-) completeness properties of the Kähler-Einstein metrics.
- §4.5: We give some applications to automorphism groups and show how to deduce the Yau-Tian-Donaldson conjecture for canonically polarized varieties from our results.
- §4.6: The paper is concluded with a brief outlook on possible applications to Miyaoka-Yau types inequalities, as well as the Weil-Peterson geometry of the moduli space of stable varieties. These applications will require a more detailed regularity analysis of the Kähler-Einstein metrics that we leave for the future.

4.1. Preliminaries

We collect here some useful tools or notions that we are going to work with in this paper. We start with a compact Kähler manifold X of dimension n, and we consider a class $\alpha \in H^{1,1}(X, \mathbb{R})$ which is big. By definition, this means that α lies in the interior of the pseudo-effective cone, so that there exists a Kähler current $T \in \alpha$, that is a current which dominates some smooth positive form ω on X. We fix θ, a smooth representative of α.

The ample locus. — An important invariant attached to α is the *ample locus* of α, denoted $\mathrm{Amp}(\alpha)$, and introduced in [**Bou04**, §3.5]. This is the largest Zariski open subset U of X such that for all $x \in U$, there exists a Kähler current $T_x \in \alpha$ with analytic singularities such that T_x is smooth in an (analytic) neighbourhood of x. Its complement, called the augmented base locus, is usually denoted by $\mathbb{B}_+(\alpha)$. In the case when $\alpha = c_1(L)$ is the Chern class of a line bundle, it is known (see e.g. [**BBP10**]) that:

$$\mathbb{B}_+(L) = \bigcap_{\substack{L=A+E \\ A \text{ ample},\, E \geqslant 0}} \mathrm{Supp}(E)$$

Currents with minimal singularities. — We will be very brief about this well-known notion, and refer e.g. to [**Bou04**, §2.8], [**BBGZ09**, §1], [**Ber09**] or [**BD12**] for more details and recent results.

By definition, if T, T' are two positive closed currents in the same cohomology class α, we say that T is less singular than T' if the local potentials φ, φ' of these currents satisfy $\varphi' \leqslant \varphi + O(1)$. It is clear that this definition does not depend on the choice of the local potentials, so that the definition is consistant. In each (pseudo-effective) cohomology class α, one can find a positive closed current $T_{\min}$ which will be less singular than all the other ones; this current is not unique in general; only its class of singularities is. Such a current will be called current with minimal singularities.

One way to find such a current is to pick $\theta \in \alpha$ a smooth representative, and define then, following Demailly, the upper envelope

$$V_\theta := \sup\{\varphi \ \theta\text{−psh}, \ \varphi \leqslant 0 \text{ on } X\}$$

Once observed that V_θ is θ-psh (in particular upper semi-continuous), it becomes clear that $\theta + dd^c V_\theta$ has minimal singularities.

Non-pluripolar Monge-Ampère operator. — In the paper [**BEGZ10**], the authors define the non-pluripolar product $T \mapsto \langle T^n \rangle$ of any closed positive $(1,1)$-current $T \in \alpha$, which is shown to be a well-defined measure on X putting no mass on pluripolar sets, and extending the usual Monge-Ampère operator for Kähler forms (or having merely bounded potentials, cf [**BT87**]). Let us note that when T is a smooth positive form ω on a Zariski dense open subset $\Omega \subset X$, then its Monge-Ampère $\langle T^n \rangle$ is simply the extension by 0 of the measure ω^n defined on Ω.

Given now a θ-psh function φ, one defines its non-pluripolar Monge-Ampère by $\mathrm{MA}(\varphi) := \langle (\theta + dd^c\varphi)^n \rangle$. Then one can check easily from the construction that the total mass of $\mathrm{MA}(\varphi)$ is less than or equal to the volume $\mathrm{vol}(\alpha)$ of the class α (cf [**Bou02**]):

$$\int_X \mathrm{MA}(\varphi) \leqslant \mathrm{vol}(\alpha)$$

A particular class of θ-psh functions that appears naturally is the one for which the last inequality is an equality. We will say that such functions (or the associated currents) have *full Monge-Ampère mass*. For example, θ-psh functions with minimal singularities have full Monge-Ampère mass (cf [**BEGZ10**, Theorem 1.16]).

Plurisubharmonic functions on complex spaces. — Here again, we just intend to give a short overview of the extension of the pluripotential theory to (reduced) complex Kähler spaces. A very good reference is [**Dem85**], or [**EGZ09**, §5] which is written in relation to singular Kähler-Einstein metric. We also refer to the preliminary parts of [**Var89**] or [**FS90**].

The data of a reduced complex space X includes the data of the sheaves of continuous and holomorphic functions. So the first object we would like to give a sense to is the sheaf $\mathscr{C}^\infty_X$ of smooth functions. It may be defined as the restriction of smooth functions in some local embeddings of X in some $\mathbb{C}^n$. One defines similarly the sheaves of smooth (p,q)-forms $\mathscr{A}^{p,q}_X$ which carry the differentials $d, \partial, \bar\partial$ satisfying the usual rules; the space of currents is by definition the dual of the space of differential forms as in the smooth case. The sheaves complexes that are induced (Dolbeault, de Rham, etc.) are however not exact in general.

Another important sheaf is the one of pluriharmonic functions. They are defined to be smooth functions locally equal to the imaginary part of some holomorphic functions. One can show (see e.g [**FS90**]) that a continuous function which is pluriharmonic on X_{reg} in the usual sense is automatically pluriharmonic on X. We denote by $\mathcal{PH}_X$ the sheaf of real-valued pluriharmonic functions on X.

Let us move on to psh functions now. There are actually two possible definitions which extend the usual one for complex manifolds. The first one, introduced by Grauert and Remmert, mimics the one in the smooth case: we will say that a function $\varphi : X \to \mathbb{R}\cup\{-\infty\}$ is plurisubharmonic if it is upper semi-continuous and if for all holomorphic map $f : \Delta \to X$ from the unit disc in $\mathbb{C}$, the function $\varphi \circ f$ is subharmonic.

We could also introduce a more local definition: a function $\varphi : X \to \mathbb{R}\cup\{-\infty\}$ is strongly plurisubharmonic if in any local embeddings $i_\alpha : X \supset U_\alpha \hookrightarrow \mathbb{C}^n$, φ is the restriction of a psh function defined an open set $\Omega_i \subset \mathbb{C}^n$ containing $i_\alpha(U_\alpha)$, if $X = \cup_\alpha U_\alpha$ is an open covering.

Clearly, a strongly psh function is also psh. Actually, Fornaess and Narasimhan [**FN80**] showed that these notions coincide: a function on X is psh if and only if it is strongly psh. On *normal* spaces one still has a Riemann extension theorem for psh functions, thanks to [**GR56**]. More precisely, if X is normal, $Y \subsetneq X$ is any proper analytic subspace, and $\varphi : X \setminus Y \to \mathbb{R} \cup \{-\infty\}$ is psh, then φ extends to a (unique) psh function on X if and only if it is locally bounded above near the points of Y, condition which is always realized if Y has codimension at least two in X. In particular, if X is normal, the data of a psh function on X is equivalent to the data of a psh function on X_{reg}.

Moreover, one can show (cf [**BEG13**, Lemma 3.6.1]) that a pluriharmonic function on X_{reg} automatically extends to a pluriharmonic function on X.

On non-normal spaces, one has to be more cautious, and it is convenient to introduce the notion of weakly psh function. Let X be a reduced complex space, and $\nu : X^\nu \to X$ its normalization. We say that a function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is weakly psh if $\nu^*\varphi = \varphi \circ \nu$ is psh. It is not hard to see that a weakly psh function φ induces a bona fide psh function on X_{reg} which is locally bounded from above near the points of X_{sing}. Conversely, any psh function on X_{reg} which is locally upper bounded extends to a weakly psh function on X. On a normal space, a weakly psh function is of course psh, but in general these notions are different: consider $X = \{zw = 0\} \subset \mathbb{C}^2$, and $\varphi(x) = 0$ or 1 according to the connected component of $x \in X$. We refer to [**Dem85**, Théorème 1.10] for equivalent characterizations of weakly psh functions and conditions on a weakly psh function that ensure that it is already psh.

Finally, one can check that a (strongly) psh function φ on a complex space X is always locally integrable with respect to the area measure induced by any local embedding of X in $\mathbb{C}^n$ (note that this is stronger than saying that φ is locally integrable on X_{reg} with respect to some volume form). Moreover, a locally integrable function φ is (almost everywhere) weakly psh is and only if it is locally bounded from above and $dd^c\varphi$ is a positive current.

Weights and Chern classes. — From now on, X will be a *normal* complex space unless stated otherwise.

The definition of a (smooth) Kähler form is rather natural: it is a smooth real $(1,1)$-form written locally as $dd^c\psi$ for some (smooth) strictly psh function ψ; equivalently this is locally the restriction of a Kähler form in a embedding in $\mathbb{C}^n$. Note that we could interpret this definition in terms of hermitian metrics on the Zariski tangent bundle of X, cf [**Var89**].

Let us now consider a line bundle L on X. A smooth hermitian metric h on L is defined as in the smooth case: using trivialisations $\tau_\alpha : L_{|U_\alpha} \xrightarrow{\simeq} U_\alpha \times \mathbb{C}$, we just ask h to be written as $h(v) = |\tau_\alpha(v)|^2 e^{-\varphi_\alpha(z)}$ where φ_α is a smooth function on U_α. We say that the data $\phi := \{(U_\alpha, \varphi_\alpha)\}$ is a weight on L, so that it is equivalent to consider a weight or an hermitian metric.

Observe that if $(g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*)$ is the cocycle in $H^1(X, \mathcal{O}_X^*)$ determined by the τ_α's (more precisely $\tau_\alpha \circ \tau_\beta^{-1}(z,v) = (z, g_{\alpha\beta} v)$), then we have necessarily $\varphi_\beta - \varphi_\alpha = \log |g_{\alpha\beta}|^2$. In particular, the forms $dd^c\varphi_\alpha$ glue to a global smooth $(1,1)$-form on X called curvature of (L,h) and denoted by $c_1(L,h)$. This forms lives naturally in the space $H^0(X, \mathscr{C}_X^\infty/\mathcal{PH}_X)$, and using the exact sequence

$$0 \longrightarrow \mathcal{PH}_X \longrightarrow \mathscr{C}_X^\infty \longrightarrow C_X^\infty/\mathcal{PH}_X \longrightarrow 0$$

one may attach to (L,h) a class $\hat{c}_1(L,h) \in H^1(X, \mathcal{PH}_X)$. It is then easy to see that this class actually does not depend on the choice of h, so we will denote it by $\hat{c}_1(L)$. If X is smooth, $H^1(X, \mathcal{PH}_X) \simeq H^{1,1}(X, \mathbb{R})$, and it is well-known that $\hat{c}_1(L)$ coincides with the image of $L \in H^1(X, \mathcal{O}_X^*)$ in $H^2(X, \mathbb{Z})$ via the connecting morphism induced by the exponential exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{e^{2i\pi\cdot}} \mathcal{O}_X^* \longrightarrow 0$$

This sequence also exists on any (even non-reduced) complex space, so that $c_1(L) \in H^2(X, \mathbb{Z})$ is well-defined; it will be more convenient for us to look at the image of $c_1(L)$ in $H^2(X, \mathbb{R})$ however. To relate it to $\hat{c}_1(L)$, we may use the following exact sequence:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}_X \xrightarrow{-2\mathrm{Im}(\cdot)} \mathcal{PH}_X \longrightarrow 0 \tag{4.1.1}$$

It is not hard to check that the connecting morphism $H^1(X, \mathcal{PH}_X) \to H^2(X, \mathbb{R})$ sends $\hat{c}_1(L)$ to $c_1(L)$ as expected.

We will also have to consider singular weights, which are by definition couples $\phi := \{(U_\alpha, \varphi_\alpha)\}$ where U_α is covering of X trivializing L, and φ_α are locally integrable on U_α, satisfying $\varphi_\beta - \varphi_\alpha = \log |g_{\alpha\beta}|^2$ on $U_\alpha \cap U_\beta$. The associated curvature current, denoted by $dd^c\phi$, is well-defined on X. The weight is said psh if the φ_α are, in which case $dd^c\phi$ is a positive current. Moreover, we can proceed as in the smooth case to attach to $dd^c\phi$ a class $\hat{c}_1(L) \in H^1(X, \mathcal{PH}_X)$ (consider ϕ as a section of the sheaf $L^1_{\mathrm{loc}}/\mathcal{PH}_X$ and use the natural exact sequence), whose image in $H^2(X, \mathbb{R})$ via the long exact sequence in cohomology induced by (4.1.1) is $c_1(L)$. Therefore when ϕ is a singular weight on L, we may say that $dd^c\phi$ is a current in $c_1(L)$.

We claim that a (possibly singular) psh weight ϕ on $L_{|X_{\mathrm{reg}}}$ - and thus a psh weight in the usual sense- automatically extends to a (unique) psh weight $\widetilde{\phi}$ on L. Indeed, by Grauert and Remmert's theorem, the φ_α's defined on $U_\alpha \cap X_{\mathrm{reg}}$ extend to a psh function $\widetilde{\varphi_\alpha}$ on the whole U_α, which is moreover defined by $\widetilde{\varphi_\alpha}(z_0) = \limsup_{X_{\mathrm{reg}} \ni z \to z_0} \varphi_\alpha(z)$. Therefore, the relation $\widetilde{\varphi_\beta} = \widetilde{\varphi_\alpha} + \log |g_{\alpha\beta}|^2$ is immediately satisfied on the whole $U_\alpha \cap U_\beta$, which proves the claim.

If we get back to the non-normal case, we can define psh weights using weakly psh functions instead of psh functions. So the general philosophy is that we always pull-back our objects to the normalization where things behave better, and the notions downstairs are defined and studied upstairs. For example, we can define on a normal variety the analogue of the non-pluripolar product and consider Monge-Ampère equations as in the smooth case (cf [**BBE**$^+$**11**, §1.1-1.2]. Then, if we write them on a non-normal variety, they have to be thought as pulled-back to the normalization.

Log canonical pairs. — Following the by now common terminology of Mori theory and the minimal model program (cf e.g. [**KM98**]), a pair (X, D) is by definition a complex normal projective variety X carrying a Weil $\mathbb{Q}$-divisor D (not necessarily effective). We will say that the pair (X, D) is a log canonical pair if $K_X + D$ (which is a priori defined as a Weil divisor) is $\mathbb{Q}$-Cartier, and if for some (or equivalently any) log resolution $\pi : X' \to X$, we have:

$$K_{X'} = \pi^*(K_X + D) + \sum a_i E_i$$

where E_i are either exceptional divisors or components of the strict transform of D, and the coefficients a_i satisfy the inequality $a_i \geqslant -1$.

4.2. Singular Kähler-Einstein metrics

4.2.1. Kähler-Einstein metrics on pairs. — In this section, we will consider log pairs (X, D) where X is a complex normal projective variety, D is a Weil divisor, and $K_X + D$ is assumed to be $\mathbb{Q}$-Cartier. We choose a psh weight ϕ_D on X_{reg} satisfying $dd^c\phi_D = [D_{|X_{\mathrm{reg}}}]$. The first definition concerns the Ricci curvature of currents:

Definition 4.2.1. — Let ω be a positive current on X_{reg}; we say that ω is admissible if it satisfies:

1. Its non-pluripolar product $\langle \omega^n \rangle$ defines a (locally) absolutely continuous measure on X_{reg} with respect to $d\mathbf{z} \wedge d\bar{\mathbf{z}}$, where $\mathbf{z} = (z_i)$ are local holomorphic coordinates.
2. The function $\log(\langle \omega^n \rangle / d\mathbf{z} \wedge d\bar{\mathbf{z}})$ belongs to $L^1_{\mathrm{loc}}(X_{\mathrm{reg}})$.

In that case, we define (on X_{reg}) the Ricci curvature of ω by setting $\mathrm{Ric}\,\omega := -dd^c \log\langle \omega^n \rangle$.

Another way of thinking of this is to interpret the positive measure $\langle \omega^n \rangle_{|X_{\mathrm{reg}}}$ as a singular metric on $-K_{X_{\mathrm{reg}}}$ whose curvature is $\mathrm{Ric}\,\omega$ by definition.

The measure e^ϕ for ϕ a weight on K_X. — In the same spirit, we will use the convenient but somehow abusive notation e^ϕ for ϕ a weight on K_X (whenever the latter is defined as a $\mathbb{Q}$-line bundle) to refer to the positive measure $e^{\varphi_{\mathbf{z}}} d\mathbf{z} \wedge d\bar{\mathbf{z}}$ defined on X_{reg} and extended by 0 to X; where $\varphi_{\mathbf{z}}$ is the expression on some trivializing chart of X_{reg} (and hence of $K_{X_{\mathrm{reg}}}$ too) of ϕ. In particular, for ϕ a psh weight on $K_X + D$, the measure $e^{\phi - \phi_D}$ can easily be pulled-back to any log resolution (π, X', D') of (X, D) (we pull it back over $X_{\mathrm{reg}} \setminus \mathrm{Supp}(D)$ and then extend it by 0), where it become $e^{\phi\circ\pi - \phi_{D'}}$.

We may now introduce the notion of (negatively curved) Kähler-Einstein metric attached to a pair (X, D):

Definition 4.2.2. — Let (X, D) be a log pair; we say that a positive admissible current ω is a Kähler-Einstein metric with negative curvature for (X, D) if:

1. $\operatorname{Ric}\omega = -\omega + [D]$ on X_{reg},
2. $\int_{X_{\text{reg}}} \langle \omega^n \rangle = c_1(K_X + D)^n$.

This conditions are sufficient to show that a Kähler-Einstein metric is a global solution of a Monge-Ampère equation. More precisely, we have the following:

Proposition 4.2.3. — *Let (X, D) be a log pair, and ω be a Kähler-Einstein metric for (X, D). Then $\phi := \log\langle \omega^n \rangle + \phi_D$ extends to X as a psh weight with full Monge-Ampère mass on $K_X + D$, solution of*

$$\langle (dd^c\phi)^n \rangle = e^{\phi - \phi_D}$$

Conversely, any psh weight ϕ on $K_X + D$ with full Monge-Ampère mass solution of the equation induces a Kähler-Einstein metric $\omega := dd^c\phi$ for (X, D).

Proof. — On X_{reg}, we have $dd^c\phi = \omega$ thus ϕ is a psh weight on $(K_X + D)_{|X_{\text{reg}}}$, and thanks to a theorem of Grauert and Remmert, it extends through X_{sing} which has codimension at least 2. Clearly, we have $\omega = dd^c\phi$ on X, and by condition 3. in the definition of a Kähler-Einstein metric, ϕ has full Monge-Ampère mass. Then by definition, the two (non-pluripolar) measures $\langle (dd^c\phi)^n \rangle$ and $e^{\phi - \phi_D}$ coincide.

For the converse, let $\omega := dd^c\phi$; clearly 1. and 3. are satisfied. Moreover, ϕ and ϕ_D are locally integrable, so that ω is admissible and $\operatorname{Ric}\omega = -dd^c(\phi - \phi_D) = -\omega + [D]$. □

This proposition shows that the different definitions of what should be a singular Kähler-Einstein metric, appearing e.g. in [**Ber11, BEGZ10, CGP11, EGZ09**] etc. coincide. Moreover, one could equally define positively curved Kähler-Einstein metrics in an equivalent way as in [**BBE$^+$11**]. In particular this objects, intrinsically defined on X, can also be seen on any log resolution in the usual way; in practice, we will most of the time work on log resolutions when dealing with existence or smoothness questions.

Note also that we could have chosen to define a Kähler-Einstein metric attached to a pair (X, D) (say satisfying $K_X + D$ ample) to be a *smooth* Kähler metric ω on $X_{\text{reg}} \setminus \operatorname{Supp}(D)$ which extends to an admissible current on X_{reg} satisfying there $\operatorname{Ric}\omega = -\omega + [D]$ and the mass condition $\left(\int_{X_{\text{reg}}} \langle \omega^n \rangle =\right) \int_{X_{\text{reg}} \setminus \operatorname{Supp}(D)} \omega^n = c_1(K_X + D)^n$.

Then, our regularity Theorem (say combined with Proposition 4.5.1) shows *a posteriori* that this definition would have coincided with Definition 4.2.2.

Let us also mention that in the case of a log smooth log canonical pair (X, D), the same proof as [**Gue12b**, Proposition 2.5] shows that the data of a negatively curved Kähler-Einstein on (X, D) is equivalent to giving an admissible current ω on $X \setminus \operatorname{Supp}(D)$ such that:

- $\operatorname{Ric}\omega = -\omega$ on $X \setminus \operatorname{Supp}(D)$,
- There exists $C > 0$ such that

$$C^{-1} dV \leqslant \prod_{a_j < 1} |s_j|^{2a_j} \cdot \prod_{a_k = 1} \left(|s_k|^2 \log^2 |s_k|^2\right) \omega^n \leqslant C dV$$

for some volume form dV on X, and where $D = \sum a_i D_i$, s_i being a defining section of D_i.

4.2.2. Kähler-Einstein metrics on stable varieties. — Stable varieties, as considered e.g. in [**KSB88, Kar00, Kol, Kov12**] are the appropriate singular varieties to look at if one wants to compactify the moduli space of canonically polarized projective varieties (cf also [**Vie95**]). Before giving the precise definition of a stable variety, we explain very briefly that notion and give the connection with Kähler-Einstein theory. In the next section, we will give a more detail account of the type of singularities involved.

So first of all, we will consider complex varieties that are Gorenstein in codimension 1 (this condition replaces regularity in codimension 1 for normal varieties) and satisfy the condition S_2 of Serre. Basically, the singularities in codimension 1 of our varieties are those of the union of two coordinate hyperplanes ("double crossing"). Now we want to recast them in the context given by the singularities of the minimal model program (MMP); so we consider such a variety X and its normalization $\nu : X^\nu \to X$. One can write $\nu^* K_X = K_{X^\nu} + D$ for some reduced divisor D called the conductor of ν; its sits above the codimension 1 component of the singular locus of X. We then say that X has semi log canonical singularities if the pair (X^ν, D) is log canonical in the usual sense. The generalization of the notion of stable curve is given by the following definition:

Definition 4.2.4. — A projective variety X is called stable if X has semi-log canonical singularities, and K_X is an ample $\mathbb{Q}$-line bundle.

There is a subtlety for the definition of K_X, but we refer to §4.2.3 for appropriate explanations. Its is actually possible to define the notion of Kähler-Einstein metric for a stable variety:

Definition 4.2.5. — Let X be a stable variety. A Kähler-Einstein metric on X is a positive admissible current ω on X_{reg} such that:

1. $\mathrm{Ric}\,\omega = -\omega$ on X_{reg},
2. $\int_{X_{\mathrm{reg}}} \langle \omega^n \rangle = c_1(K_X)^n$.

In the non-normal case however, psh weight do not automatically extend across the singularities, so that it is not clear that the Kähler-Einstein metric will extend as a positive current on K_X satisfying the usual Monge-Ampère equation *globally*. Actually, this is the case as shows the following proposition:

Proposition 4.2.6. — *Let X be a stable variety, and ω a Kähler-Einstein metric on X. Then the weight $\phi := \log \omega^n$ extends to X as a weakly psh weight on K_X solution of the Monge-Ampère equation $\langle (dd^c\phi)^n \rangle = e^\phi$.*

Proof. — Taking the dd^c of each side in the definition of ϕ and using the Ricci equation, we find $\omega = dd^c\phi$ on X_{reg}, and therefore ϕ satisfies $\langle (dd^c\phi)^n \rangle = \langle \omega^n \rangle = e^\phi$. Pulling back this equation to normalization X^ν, we find a psh weight $\phi' = \nu^*\phi$ on $c_1(\nu^* K_X)_{|\nu^{-1}(X_{\mathrm{reg}})}$ solution of $\langle (dd^c\phi')^n \rangle = e^{\phi' - \phi_D}$ where D is the conductor of the normalization. As we work inside X^ν_{reg} and the integral $\int_{\nu^{-1}(X_{\mathrm{reg}})} e^{\phi' - \phi_D}$ is finite, we infer from Lemma 4.2.7 below that ϕ'

extends (as a psh weight) across D_{reg}. So ϕ' induces a psh weight on $c_1(\nu^* K_X)_{|X^\nu_{\mathrm{reg}} \setminus D_{\mathrm{sing}}}$, and by normality of X^ν, it extends to the whole X^ν, which means precisely that ϕ extends as a weakly psh weight on K_X. The expected Monge-Ampère equation holds automatically on X (or equivalently on X') since both measures $\langle (dd^c\phi)^n \rangle$ and e^ϕ put no mass on X_{sing} by definition. □

In the previous proof, we used the following extension result:

Lemma 4.2.7. — *Let U be a neighbourhood of $0 \in \mathbb{C}^n$, $H = \{z_1 = 0\} \subset \mathbb{C}^n$, and φ be a psh function defined on $U \setminus H$. We assume that the integral*

$$\int_{U\setminus H} \frac{e^\varphi}{|z_1|^2} dV$$

is finite. Then φ extends across H, and more precisely φ tends to $-\infty$ near H.

Proof. — (thanks to Bo Berndtsson for providing us with this elegant proof) Assume, to get a contradiction, that φ does not tend to $-\infty$ near H, and let $V := U \setminus H$. Then we can find a sequence (x_k) of points in V converging to H such that $\varphi(x_k) \geqslant -C$ for some constant C. Moreover, one may assume that $x_k = (t_1^k, \ldots, t_n^k)$ converges to 0, and that $|t_1^{k+1}| < |t_1^k|/3$. Thus we can find some polydiscs D_k centered at x_k, with polyradius $(r_k, \delta, \ldots, \delta)$ such that $r_k = |t_1^k|/2$, and $\delta > 0$ is fixed; in particular a simple computations shows that no two of the D_k meet. Now,using the mean value inequality for φ at x_k, we find:

$$-C \leqslant \frac{1}{\mathrm{vol}(D_k)} \int_{D_k} \varphi \, dV$$

Therefore, using Jensen's inequality, we obtain, up to modifying C by a normalization factor depending only on the dimension n:

$$e^{-C} \leqslant \int_{D_k} \frac{e^\varphi dV}{r_k^2 \delta^{2(n-1)}}$$

but on D_k, $|z_1| \leqslant 3r_k$ so

$$e^{-C'} \leqslant \int_{D_k} \frac{e^\varphi dV}{|z_1|^2}$$

for $C' = C + \log 9 - 2(n-1)\log\delta$. As the D_k's are disjoint, it shows that the integral $\int_{U\setminus H} \frac{e^\varphi}{|z_1|^2} dV$ is infinite, which is absurd. □

4.2.3. Singularities of stable varieties. — In this paragraph, we intend to give a more precise overview of the notion of semi-log canonical singularities. As we will just touch on this topic, we refer to the nice survey [**Kov12**] for a broader study. Other good references are [**Kol, KSB88**].
In the following, X will always be a reduced and equidimensional scheme of finite type over $\mathbb{C}$, and we set $n := \dim X$.

The conditions G_1 and S_2. — As we saw earlier, we need a canonical sheaf. The condition G_1 will guarantee its existence, and the condition S_2 will (among other things) ensure its uniqueness.

If X is Cohen-Macaulay (for every $x \in X$, the depth of $\mathcal{O}_{X,x}$ is equal to its Krull dimension), then X admits a dualizing sheaf ω_X. We say that X is Gorenstein if X is Cohen-Macaulay and ω_X is a line bundle. We say that X is G_1 if X is Gorenstein in codimension 1, which means that there is an open subset $U \subset X$ which is Gorenstein and satifies $\mathrm{codim}_X(X \setminus U) \geqslant 2$.

We say that X satisfies the condition S_2 of Serre if for all $x \in X$, we have $\mathrm{depth}(\mathcal{O}_{X,x}) \geqslant \min\{\mathrm{ht}(\mathfrak{m}_{X,x}), 2\}$, where $\mathrm{ht}(\mathfrak{m}_{X,x}) = \mathrm{codim}(\bar{x})$ denotes the height of the maximal ideal $\mathfrak{m}_{X,x}$ of $\mathcal{O}_{X,x}$.This condition is equivalent to saying that for each closed subset $i : Z \hookrightarrow X$ of codimension at least two, the natural map $\mathcal{O}_X \to i_*\mathcal{O}_{X\setminus Z}$ is an isomorphism.

If X is G_1 and S_2, and $U \subset X$ is a Gorenstein open subset whose complement has codimension at least 2, one can then define the canonical sheaf ω_X by $\omega_X := j_*\omega_U$ where $j : U \hookrightarrow X$ is the open embedding, and ω_U is the dualizing sheaf of U. By definition, this is a rank one reflexive sheaf. When X is projective, we know that it admits a dualizing sheaf; as it is reflexive, it coincides with ω_X by the S_2 condition.

We would like to have an interpretation of ω_X, or at least ω_U in terms of Weil divisor as in the normal case where we define the Weil divisor K_X as the closure of some Weil divisor representing the line bundle $K_{X_{\mathrm{reg}}}$. But we have to be more cautious in the non normal case it is not clear how we should extend a Weil divisor given on X_{reg}. Actually, this is where the G_1 conditions appears: as ω_U is a line bundle, or equivalently a Cartier divisor, we may choose a Weil divisor K_U whose support does not contain any component of X_{sing} of codimension 1 and represent ω_U (write ω_U as the difference of two very ample line bundles). Then we define K_X to be the closure of K_U. Clearly, the divisorial sheaf $\mathcal{O}_X(K_X)$ is reflexive, and coincides with $\omega_U = \omega_{X|U}$ on U, so that by the S_2 condition, we get:

$$\omega_X \simeq \mathcal{O}_X(K_X)$$

In fact, if $\omega_X^{[m]}$ denotes the m-th reflexive power of ω_X, the same arguments yield $\omega_X^{[m]} \simeq \mathcal{O}_X(mK_X)$. Therefore, the Weil divisor is $\mathbb{Q}$-Cartier if and only if ω_X is a $\mathbb{Q}$-line bundle, ie $\omega_X^{[m]}$ is a line bundle for some $m > 0$.

Conductors and slc singularities. — Let now X be a (reduced) scheme, and $\nu : X^\nu \to X$ its normalization. We recall that if X is not irreducible, its normalization is defined to be the disjoint union of the normalization of its irreducible components. The *conductor ideal*

$$\mathrm{cond}_X := \mathscr{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X)$$

is the largest ideal sheaf on X that is also an ideal sheaf on X^ν. If we think of the case where B is the integral closure of some integral ring A, then we can easily see that $\mathrm{Hom}_A(B, A)$ injects in A (via the evaluation at 1), and the image of this map is the annihilator $\mathrm{Ann}_A(B/A) = \{f \in A; fB \subset A\}$, or equivalently the largest ideal $\mathcal{I} \subset A$ that is also an ideal in B.

Coming back to the case of varieties, we will denote by cond_{X^ν} the conductor seen as an ideal sheaf on X^ν, and we define the conductor subschemes as $C_X := \mathrm{Spec}_X(\mathcal{O}_X/\mathrm{cond}_X)$ and $C_{X^\nu} := \mathrm{Spec}_{X^\nu}(\mathcal{O}_{X^\nu}/\mathrm{cond}_{X^\nu})$. If X is S_2, then one can show that these schemes have pure codimension 1 (and hence define Weil divisors) but they are in general not reduced (e.g. the cusp $y^2 = x^3$).

If K_X is $\mathbb{Q}$-Cartier and X is demi-normal (ie X is S_2 and has only double crossing singularities in codimension 1, cf [**Kol**]), we have the following relation:

$$\nu^* K_X = K_{X^\nu} + C_{X^\nu} \tag{4.2.1}$$

The proof of this identity goes as follows: first, using the demi-normality assumption, we may assume that the only singularities of X are double normal crossings. Then, using the universal property of the dualizing sheaf (which coincide with the canonical sheaf as we observed above) and the projection formula, we have $\nu_* \omega_{X^\nu} = \omega_X(-C_X)$. We pull-back this relation to X^ν using the fact that the sheaf $\mathcal{O}_X(-C_X)$ becomes precisely $\mathcal{O}_{X^\nu}(-C_{X^\nu})$. By the assumptions on the singularities, this last sheaf is actually an invertible sheaf so that we get the expected identity (cf point 8 in [**Kol**]). As we will explain below, we do not want to assume a priori that our varieties are demi-normal. Therefore, it may happen that C_{X^ν} is not Cartier, and the formula (4.2.1) may not be true anymore. So whenever we will deal with Kähler-Einstein on those varieties, we will have to apply the arguments on a log-resolution of the normalization instead of the normalization itself. Anyway, this will not cause any troubles.

An important point is that whenever the conductor is reduced, then necessarily X is seminormal (ie every finite morphism $X' \to X$ (with X' reduced) that is a bijection on points is an isomorphism); moreover, a seminormal scheme which is G_1 and S_2 has only double crossing singularities in codimension 1, ie it is demi-normal. We refer to [**Tra70, GT80, KSS10**]) for the previous assertions. This leads to the following definition:

Definition 4.2.8. — We will say that X has semi-log canonical singularities if:

1. X is G_1 and S_2,
2. K_X is $\mathbb{Q}$-Cartier,
3. The pair (X^ν, C_{X^ν}) is log-canonical.

If X has semi-log canonical singularities (slc), then C_{X^ν} is necessarily reduced, and therefore the codimension 1 singularities of X are only double crossing as we explained above. This assumption is usually added in the definitions (cf [**Kol, Kov12**]), but we may keep it or not without any change. This justifies the seemingly different definition given in the previous section. Finally, we can give the definition of a stable variety:

Definition 4.2.9. — We say that X is stable if

1. X is projective,
2. X has semi-log canonical singularities,
3. K_X is $\mathbb{Q}$-ample.

Singularities and Kähler-Einstein metrics: a summary. — If we take a closer look at the proof of Proposition 4.2.6, we see that we did not use all of the properties of a stable variety to see that a Kähler-Einstein metric always extend. Actually, we just used the fact that the conductor was a divisor. Therefore, using the existence and regularity results that we are going to prove in the next sections, and the restriction on the singularities of a pair carrying a Kähler-Einstein metric (cf Proposition 4.5.1), we can summarize the problem of the existence of a Kähler-Einstein metric on a stable variety in the following statement:

Theorem 4.2.10. — *Let X be reduced n-equidimensional projective scheme of finite type over $\mathbb{C}$, satisfying the conditions G_1 and S_2, and such that K_X is an ample $\mathbb{Q}$-line bundle. Then the following are equivalent:*

1. *There exists a Kähler form ω on X_{reg} such that $\mathrm{Ric}\,\omega = -\omega$ and $\int_{X_{\mathrm{reg}}} \omega^n = c_1(K_X)^n$,*
2. *There exists ω as above which extends to define a positive current in $c_1(K_X)$,*
3. *X has semi-log canonical singularities, ie X is stable.*

Moreover, by the results of Odaka [**Oda08, Oda11**], *the latter condition is equivalent to:*

4. *The pair (X, K_X) is K-stable.*

4.3. Variational solutions

4.3.1. General setting. — Consider the following general setting: X is a compact Kähler manifold and $[\omega]$ a big class, with ω smooth (but not necessarily positive). We say that a function $u \in PSH(X,\omega)$ has *full Monge-Ampère mass,* and we will write $u \in \mathcal{E}(X,\omega)$, if the total mass of $\mathrm{MA}(u)$ is equal to the volume of the class $[\omega]$, where the volume in question may be defined by $V := \mathrm{vol}([\omega]) := \int_X \mathrm{MA}(u_{\min})$, for $u_{\min}$ any element in $PSH(X,\omega)$ with minimal singularities, cf §4.1.

We now recall an important subspace of $\mathcal{E}(X,\omega)$ denoted by $\mathcal{E}^1(X,\omega)$, and consisting of functions with finite energy. The energy $\mathcal{E}(u)$ of an ω-psh function u (not necessarily in $\mathcal{E}(X,\omega)$) is defined in the following way (cf [**GZ07, BEGZ10, BBGZ09**] for more details – the energy is sometimes denoted by E in the aforementioned papers).
First, if $u \in PSH(X,\omega)$ has minimal singularities, we set

$$\mathcal{E}(u) := \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X (u - V_\theta)\, \mathrm{MA}(u^{(j)}, V_\theta^{(n-j)})$$

where MA is the mixed non-pluripolar Monge-Ampère operator. If now u is any ω-psh function, we defined

$$\mathcal{E}(u) := \inf \{\mathcal{E}(v) \,|\, v \in PSH(X,\omega) \,\text{with minimal singularities,}\ v \geqslant u\}$$

Then we set $\mathcal{E}^1(X,\omega) := \{u \in PSH(X,\omega),\, \mathcal{E}(u) > -\infty\}$. Actually, [**BEGZ10**, Proposition 2.11] gives another characterization of this last space: a function $u \in PSH(X,\omega)$ belongs to $\mathcal{E}^1(X,\omega)$ if and only if $u \in \mathcal{E}(X,\omega)$ and $\int_X (u - V_\theta)\mathrm{MA}(u) < +\infty$ (and for any $u \in \mathcal{E}(X,\omega)$, the explicit integral formula for $\mathcal{E}(u)$ above is still valid). Using this result, it becomes clear that $\mathcal{E}^1(X,\omega) \subset \mathcal{E}(X,\omega)$ as announced.

We should finally add that $\mathcal{E}$ is an upper-semicontinuous (usc) concave functional on $PSH(X,\omega)$, and that it is the normalized primitive of the Monge-Ampère operator, i.e.

$$(d\mathcal{E})_u = \frac{1}{V}\, \mathrm{MA}(u)^n. \tag{4.3.1}$$

4.3.2. Uniqueness. — Given a measure μ on X (possible non-finite) we consider the following MA-equation for $u \in PSH(X,\omega)$ attached to the pair (ω,μ) :

$$\omega_u^n = e^u \mu, \tag{4.3.2}$$

where $\omega_u^n := \mathrm{MA}(u)$ is the non-pluripolar Monge-Ampère operator as before. This equation is equivalent to the following *normalized* MA-equation on $\mathcal{E}(X,\omega)/\mathbb{R}$:

$$\frac{\omega_u^n}{V} = \frac{e^u\mu}{\int e^u\mu}, \tag{4.3.3}$$

The equivalence follows immediately from the $\mathbb{R}-$invariance of the latter equation and the substitution $u \mapsto u - \log \int e^u\mu$ which maps solutions of equation (4.3.2) to solutions of the equation (4.3.3).

Proposition 4.3.1. — *Any two solutions u and v of the MA-equation* (4.3.2) *such that u and v are in $\mathcal{E}(X)$ coincide.*

Proof. — This is an immediate consequence of the comparison principle [**BEGZ10**, Corollary 2.3]: if u and v are in $\mathcal{E}(X)$ then

$$\int_{\{u<v\}} \mathrm{MA}(v) \leqslant \int_{\{u<v\}} \mathrm{MA}(u)$$

But the MA above then forces $u = v$ a.e wrt the measure μ. Since μ cannot charge pluripolar sets (as $\mathrm{MA}(u)$ does not) it follows that $u = v$ away from a pluripolar set and hence everywhere, by basic properties of psh functions. □

4.3.3. Existence results for log canonical pairs. — Let (X,D) be a log canonical pair such that the log canonical divisor $K_X + D$ is big. Assume that (X,D) is a log smooth, i.e. X is smooth and

$$D = \sum_i c_i D_i$$

is a normal crossings divisor with $c_i \in]-\infty,1]$. To the pair (X,D) we can associate the following Kähler-Einstein type equation for a metric ϕ on $L := K_X + D$:

$$(dd^c\phi)^n = e^{\phi-\phi_D}, \tag{4.3.4}$$

where $\phi_D = \sum_i c_i \log |s_i|^2$ and s_i are sections cutting out the divisors D_i above.

Theorem 4.3.2. — *There is a unique finite energy solution ϕ to the equation above.*

Proof. — The proof is a modification of the variational approach in [**BBGZ09**] (concerning the case when D is trivial). To explain this we fix a smooth form $\omega \in c_1(K_X + D)$. Then the equation above is equivalent to a Monge-Ampère equation for an $\omega-$psh function u :

$$\omega_u^n = e^u\mu \tag{4.3.5}$$

where the measure μ is of the form $\mu = \rho dV$ for a function ρ in $L^{1-\delta}(X)$ (but ρ is not in $L^1(X)$!). We let

$$\mathcal{L}(u) := -\log \int e^u\mu$$

Then, at least formally, solutions of equation (4.3.5) are critical points of the functional

$$\mathcal{G}(u) := \mathcal{E}(u) + \mathcal{L}(u).$$

in view of the equation (4.3.1) satisfied by $\mathcal{E}$. $\mathcal{L}$ also defines an usc concave functional on $PSH(X,\omega)$ and we let $\mathcal{L}(X,\omega) := \{\mathcal{L} > -\infty\}$ (the upper semi-continuity follows from Fatou's lemma).

Note that Lemma 4.3.3 below guarantees that the intersection $\mathcal{E}^1(X,\omega)\cap\mathcal{L}(X,\omega)$ is non-empty. Hence, $\mathcal{G}(u)$ is not identically equal to $-\infty$ on its domain of definition that we will take to be $\mathcal{E}^1(X,\omega)$ (equipped with the usual $L^1(X)-$topology).

Next, we observe that

$$\mathcal{G}(u) \leqslant \mathcal{E}(u) - \int u\mu_0 + C'' \tag{4.3.6}$$

Indeed, since $\mu \geqslant C\mu_0$, where μ_0 is finite measure on X integrating all quasi-psh functions on X (in our case we may take $\mu_0 = \|s'\|\, dV$ for some holomorphic section s' defined by the negative coefficients of D) :

$$\int e^u\mu \geqslant C\int e^u\mu_0$$

and hence

$$\mathcal{L}(u) \leqslant C' - \log\int e^u\mu_0 \leqslant C'' - \int u\mu_0$$

using Jensen's inequality, which proves (4.3.6). In particular, $\mathcal{G}(u)$ is bounded from above. Indeed, by scaling invariance we may assume that $\sup_X u = 0$ and then use that, by basic compactness properties of $\omega-$psh functions, $\sup u \leqslant \int u\mu_0 + C$.

Let now $u_j \in \mathcal{E}^1(X,\omega)$ be a sequence such that

$$\mathcal{G}(u_j) \to \sup_{\mathcal{E}^1(X,\omega)} \mathcal{G} := S < \infty$$

Again, by the scale invariance of $\mathcal{G}$ we may assume that $\sup_X u_j = 0$. In particular,

$$\mathcal{L}(u_j) \geqslant S/2 - \mathcal{E}(u_j)$$

for $j > j_0$. But, by (4.3.6), $\mathcal{E}(u_j)$ is bounded from below and hence there is a constant C such that

$$\mathcal{E}(u_j) \geqslant -C,\ \ \mathcal{L}(u_j) \geqslant -C$$

Let now u_* be a limit point of u_j. By upper-semicontinuity we have that

$$\mathcal{E}(u) \geqslant -C,\ \ \mathcal{L}(u) \geqslant -C$$

Finally, we note that u_* satisfies the equation (4.3.5) by applying the projection argument from [**BBGZ09**] as follows. Fixing $v \in \mathcal{C}^\infty(X)$ let $f(t) := \mathcal{E}_\omega(P_\omega(u_* + tv)) + \mathcal{L}(u_* + tv)$, where

$$P_\omega(u)(x) := \sup\{v(x) : \ v \leqslant u,\ v \in PSH(X,\omega)\}$$

(note that $f(t)$ is finite for any t). The functional $\mathcal{L}(u)$ is decreasing in u and hence the sup of $f(t)$ on $\mathbb{R}$ is attained for $t = 0$. Now $\mathcal{E}_\omega \circ P_\omega$ is differentiable with differential $MA(P_\omega u)$ at u [**BBGZ09**]. Hence, the condition $df/dt = 0$ for $t = 0$ gives that the variational equation (4.3.5) holds when integrated against any $v \in \mathcal{C}^\infty(X)$. □

Let us now prove the following result, that we used in the proof:

Lemma 4.3.3. — *Let (X, D) be a log smooth pair and L a big line bundle. Let θ be a smooth $(1,1)$ form whose cohomology class is $c_1(L)$. Let s_0 be a section of D, and $|\cdot|$ a smooth hermitian metric on $\mathcal{O}_X(D)$. Then there exists a θ-psh function $u \in \mathcal{E}^1(X,\theta)$ such that $e^u/|s_0|^2$ is integrable.*

Proof. — As L is big, the exists m big enough such that $mL - D$ is effective. We choose t a holomorphic section of $mL - D$, and consider $s := s_0 t$ which is a section of mL vanishing along D. Let h_0 be an smooth hermitian metric on L with curvature form θ, and let V_θ be the upper envelope of all (normalized) θ-psh functions. We define on mL the hermitian metric $h := h_0^{\otimes m} e^{-mV_\theta}$. For $0 < \alpha < 1$ small enough we claim that the function

$$u := V_\theta - \left(-\frac{1}{m}\log|s|_h^2\right)^\alpha$$

suits our requirements.

First of all, it is θ-psh because of the following general fact: if ψ is θ-psh and $\chi : \mathbb{R} \to \mathbb{R}$ is convex and non-decreasing satisfying $\chi' \leqslant 1$, then $V_\theta + \chi(\psi - V_\theta)$ is θ-psh. Indeed, $dd^c(V_\theta + \chi(\psi - V_\theta)) = (1-\chi')(\theta + dd^c V_\theta) + \chi'(\theta + dd^c\psi) + \chi''|d(\psi - V_\theta)|^2$ where χ' and χ'' are evaluated at $\psi - V_\theta$. Now we apply this to $\psi = 1/m \log|s|^2_{h_0^m}$.

For the integrability property, we use the following inequality for x a real number (big enough): $x^\alpha \geqslant (n+1)\log x - C$ for some $C > 0$ depending only on α. Now we observe that $V_\theta + \chi(\psi - V_\theta) \leqslant \chi(\psi)$: indeed, $\chi(\psi - V_\theta) - \chi(\psi) \leqslant \sup \chi' \cdot (-V_\theta) \leqslant -V_\theta$, so that in our case, $u \leqslant (-\frac{1}{m}\log|s|^2_{h_0})^\alpha$. If we apply the basic inequality stated above to $x = -\frac{1}{m}\log|s|^2_{h_0}$ which can be made big enough by multiplying h_0 by a big constant (this does not change the curvature), we get

$$e^u \leqslant C\left(-\frac{1}{m}\log|s|^2_{h_0}\right)^{-(n+1)}$$

As D has snc support, and $|t|$ is bounded from above, we are left to check that the integral

$$\int_D \frac{dV}{\prod_{i\leqslant n}|z_i|^2 \cdot \log^{n+1}(\prod_{i\leqslant n}|z_i|^2)}$$

over the unit polydisc D in $\mathbb{C}^n$ converges. But after a polar change of coordinate, we are led to estimate $\int_{[0,1]^n} \frac{dx_1\cdots dx_n}{\prod_{i\leqslant n} x_i \cdot \log^{n+1}(\prod_{i\leqslant n} x_i^2)}$, which equals $\frac{1}{2^{n+1}n}\int_{[0,1]^{n-1}} \frac{dx_1\cdots dx_{n-1}}{\prod_{i\leqslant n-1} x_i \cdot \log^n(\prod_{i\leqslant n-1} x_i)}$. By induction, and using the Poincaré case, it concludes.

Finally, one has to check that $u \in \mathcal{E}^1(X,\theta)$. We compute the capacity $\mathrm{Cap}_\theta(u < V_\theta - t)$ for t big. But $(u < V_\theta - t) = \left(\frac{1}{m}\log|s|^2_h < -t^{1/\alpha}\right) \subset \left((\frac{1}{m}\log|s|^2_{h_0} < -t^{1/\alpha})\right)$ and thus $\mathrm{Cap}_\theta(u < V_\theta - t) \leqslant \frac{C}{t^{1/\alpha}}$ because for every θ-psh function ψ, one has $\mathrm{Cap}_\theta(\psi < -t) \leqslant \frac{C_\psi}{t}$ (this is an easy generalization of [**GZ05**, Proposition 2.6]). Therefore, if $\alpha < \frac{1}{n+1}$, one has

$$\int_0^{+\infty} t^n \mathrm{Cap}_\theta(u < V_\theta - t)dt < +\infty$$

which, using the characterization given in [**BBGZ09**, Lemma 2.9], ends the proof of the lemma. □

Remark 4.3.4. — The proof of the preceding lemma yields actually a stronger result. If $\sum a_i \mathrm{div}(s_i)$ is an effective divisor with snc support meeting D transversally and such that $a_i < 1$ for all i, then the function u obtained above satisfies $e^u / \prod |s_i|^{2a_i} |s_0|^2 \in L^1(dV)$, and more generally this is still true for $e^{\varepsilon u}$ for all $\varepsilon > 0$ (use the inequality $\varepsilon x^\alpha \geqslant (n+1)\log x - C$ for $x = -\frac{1}{m}\log |s|^2_{h_0}$ this time).

4.3.4. Stability under perturbations. — Let now L be a semipositive and big line bundle, and consider the perturbed ample lind bundles $L_j := L + \varepsilon_j A$, for ε_j a sequence of positive numbers tending to 0 and A a fixed ample line bundle. Fixing also a Kähler form $\omega_A \in c_1(A)$ and a smooth semipositive form $\omega \in c_1(L)$, we write $\omega_j := \omega + \varepsilon_j \omega_A$. Let μ_j be the sequence of measures on X given by

$$\mu_j = \prod_\alpha (|s_\alpha|^2 + \varepsilon_j)^{e_\alpha} \frac{dV}{\prod_\beta |s_\beta|^2}$$

where $e_\alpha > -1$ for all α, and the divisor $\sum_\alpha \mathrm{div}(s_\alpha) + \sum_\beta \mathrm{div}(s_\beta)$ is a reduced normal crossing divisor. This is precisely the sequence of approximations we are going to use to solve our Kähler-Einstein equation.
Consider now the following Monge-Ampère equations for $u_j \in \mathcal{E}(X, \omega_j)$ (and sup-normalized):

$$\omega^n_{u_j}/V = \frac{e^{u_j}\mu_j}{\int_X e^{u_j}\mu_j}$$

and similarly

$$\omega^n_u/V = \frac{e^u \mu}{\int_X e^u \mu}$$

for $u \in \mathcal{E}(X, \omega)$.

Theorem 4.3.5. — *The unique sup-normalized solution u_j of the first equation above converges, in the $L^1(X)$−topology, to the unique sup-normalized solution u to the the latter equation. Equivalently, the solutions v_j of the corresponding non-normalized equations converge in $L^1(X)$ to v solving the corresponding limiting non-normalized equation.*

Proof. — We denote by $\mathcal{G}_j$ (resp. $\mathcal{L}_j$) the functional determined by the pair (ω_j, μ_j) (resp. μ_j), and by u_j the sup-normalized maximizer of $\mathcal{G}_j$. We also denote by u_0 the sup-normalized fixed ω-psh function given by Lemma 4.3.3. Let us add that in the course of the proof, the precise value of the constant C may, as usual, change from line to line. We split the proof into four steps.

Step 1. We first show that

$$-C \leqslant \mathcal{G}_j(u_j) \leqslant C \tag{4.3.7}$$

As u_0 is ω-psh, it is also ω_j-psh. Moreover, the capacity computation of Lemma 4.3.3 shows that the energy of u_0 *with respect to* ω_j is finite, and as $\mathcal{E}_{\omega_j}(u_0)$ increases with j, we obtain

$$\mathcal{E}_{\omega_j}(u_0) \geqslant -C$$

Besides, by dominated convergence, we have $\lim_{j\to+\infty} \mathcal{L}_j(u_0) = \mathcal{L}(u_0)$ and therefore we get $L_j(u_0) \geqslant -C$. Consequently, $\mathcal{G}_j(u_j) \geqslant \mathcal{G}_j(u_0) \geqslant -C$ which gives a first bound (recall that u_j maximize $\mathcal{G}_j$ by Theorem 4.3.2 and the translation invariance of $\mathcal{G}_j$).

Choose now a probability measure μ_0 satisying $\mu_j \geqslant e^{-C}\mu_0$ for all j (its existence is clear given the precise form of μ_j). Then Jensen's inequality gives

$$\mathcal{L}_j(u_j) \leqslant -\int_X u_j d\mu_0 - C$$

but the compactness properties of quasi-psh functions (all u_j's are $C\omega_A$-psh) also gives the inequality

$$\sup u_j = 0 \leqslant \int_X u_j d\mu_0 + C$$

Combining the two previous inequalities, we get

$$\mathcal{G}_j(u_j) \leqslant \mathcal{E}_{\omega_j}(u_j) + C$$

which gives both the uniform upper bound for $\mathcal{G}_j(u_j)$ (as $\mathcal{E}_{\omega_j}$ is always non-positive) and a uniform lower bound $\mathcal{E}_{\omega_j}(u_j) \geqslant -C$.

Let u be an L^1-limit point in $PSH(X, \omega)$ of the sequence (u_j).

Step 2. We would now like to see that

$$\mathcal{G}(u) \geqslant \limsup \mathcal{G}_j(u_j) \tag{4.3.8}$$

The part involving the $\mathcal{L}$ functional is easy: indeed, by Fatou's lemma, we have

$$\mathcal{L}(u) \geqslant \limsup \mathcal{L}_j(u_j)$$

As for the energy part, things get more complicated. We also have

$$\mathcal{E}(u) \geqslant \limsup \mathcal{E}_{\omega_j}(u_j)$$

but we will give the proof later in Lemma 4.3.6. Putting this two inequalities together (and using the basic inequalities relating the limsup of a sum to the sum of the limsup), we get

$$\mathcal{G}(u) \geqslant \limsup \mathcal{G}_j(u_j)$$

Step 3. Let us show that u is a sup-normalized maximizer of $\mathcal{G}$.

Here again, this is quite subtle; let us stress out why. We choose a sup-normalized ω-psh function v and we want to show that

$$\mathcal{G}(u) \geqslant \mathcal{G}(v) \tag{4.3.9}$$

Of course, on can assume that $\mathcal{G}(v)$ is finite. Thanks to step 2, it is enough to show that $\limsup \mathcal{G}_j(v) \geqslant G(v)$. But this inequality is far from clear as we cannot directly apply the dominated convergence theorem here. Indeed, for the energy part, it could happen that $\mathcal{E}_\omega(v)$ is finite though $\mathcal{E}_{\omega_j}(v) = -\infty$ for all j. As for the other part, despite $e^v \in L^1(\mu)$, it is not obvious that $e^v \in L^1(\mu_j)$ (because of the "zeroes" of μ which do not appear in μ_j).

To bypass these difficulties, we proceed to some sort of regularization enabling to deal both issues. More precisely, we pick a family of smooth ω-psh functions $(v_\delta)_{\delta>0}$ which decreases to v, and we set for all positive δ, ε:

$$v_{\delta,\varepsilon} := (1-\varepsilon)v_\delta + \varepsilon u_0$$

where we recall that u_0 denotes the particular (sup-normalized) ω-psh function constructed in Lemma 4.3.3.

As v_δ is smooth and $u_0 \in \mathcal{E}^1(X,\omega_j)$ for all j, $v_{\delta,\varepsilon}$ has finite ω_j-energy, the dominated convergence theorem shows that

$$\lim_{j\to+\infty} \mathcal{E}_{\omega_j}(v_{\delta,\varepsilon}) = \mathcal{E}_\omega(v_{\delta,\varepsilon}) \tag{4.3.10}$$

Moreover, as we observed in remark 4.3.4, the function $e^{\varepsilon u_0}$ is in $L^1(\mu)$ for all $\varepsilon > 0$, and $e^{v_{\delta,\varepsilon}} \leqslant e^{\varepsilon u_0}$. Therefore, by dominated convergence, we get

$$\lim_{j\to+\infty} \mathcal{L}_j(v_{\delta,\varepsilon}) = \mathcal{L}(v_{\delta,\varepsilon}) \tag{4.3.11}$$

Combining (4.3.8) with (4.3.10) and (4.3.11), we get $\mathcal{G}(u) \geqslant \mathcal{G}(v_{\delta,\varepsilon})$ for all $\delta,\varepsilon > 0$. Set $v_\varepsilon := (1-\varepsilon)v + \varepsilon u_0$. By monotonicity of $\mathcal{E}_\omega$, we have $\mathcal{E}_\omega(v_{\delta,\varepsilon}) \geqslant \mathcal{E}_\omega(v_\varepsilon)$. Using the dominated convergence theorem, we also see that $\mathcal{L}(v_{\delta,\varepsilon}) \to \mathcal{L}(v_\varepsilon)$. Therefore, we have $\mathcal{G}(u) \geqslant \mathcal{G}(v_\varepsilon)$. Finally, using the concavity of $\mathcal{G}$, we get $\mathcal{G}(u) \geqslant (1-\varepsilon)\mathcal{G}(v) + \varepsilon\mathcal{G}(u_0)$, and we get (4.3.9) by letting ε go to zero.

Step 4. Back to the non-normalized equation. We have $v_j = u_j + \mathcal{L}_j(u_j)$ and as shown above in (4.3.7), $\mathcal{G}_j(u_j)$ is a bounded sequence (more precisely, it converges to the maximal value S of $\mathcal{G}$) and $0 \leqslant -\mathcal{E}(u_j) \leqslant C$, which implies that $\mathcal{L}_j(u_j)$ is also a bounded sequence. After passing to a subsequence we may thus assume that $\mathcal{L}_j(u_j) \to l \in \mathbb{R}$ so that $v_j \to v := u + l$, solving the desired equation (and $v \in \mathcal{E}^1(X,\omega)$). By the uniqueness of solutions of the latter equation this means that the whole sequence u_j converges to v, which concludes the proof. □

Let us now give the proof of the following result which was essential for step 2:

Lemma 4.3.6. — *Let $[\omega_j]$ and $[\omega]$ be semi-positive big classes such that $\omega_j \to \omega$ in the $\mathcal{C}^\infty-$topology of smooth (semipositive) forms. If $u_j \in \mathcal{E}(X,\omega_j)$ (and $u \in \mathcal{E}(X,\omega)$) such that $u_j \to u$ in $L^1(X)$, then*

$$\mathcal{E}_\omega(u) \geqslant \limsup_j \mathcal{E}_{\omega_j}(u_j)$$

Proof. — When $\omega_j = \omega$ the lemma amounts to the well-known fact that $\mathcal{E}_\omega$ is usc. We may as well assume that $\sup u_j = \sup u = 0$.

First of all, we modify the sequence (u_j) to make it non-increasing. More precisely, we set $\tilde{u}_j := (\sup_{k\geqslant j} u_k)^*$, which defines an ω_j-psh function. Then $\tilde{u}_j \geqslant u_j$ and the sequence $(\tilde{u}_j)_j$ is non-increasing. Given v an ω-psh function and $c \in \mathbb{R}$, we will write $v^c := \max\{v, c\}$. By construction $\tilde{u}_j^c$ decreases to u^c, and all these functions are ω_0-psh. By the local convergence result of Bedford-Taylor for mixed Monge-Ampère expressions and the smooth convergence of ω_j to ω, we see that

$$\mathcal{E}_\omega(u^c) = \lim_{j\to+\infty} \mathcal{E}_{\omega_j}(\tilde{u}_j^c)$$

As $u_j \leqslant \tilde{u}_j \leqslant \tilde{u}_j^c$, the monotonicity of $\mathcal{E}_\omega$ ensures that

$$\mathcal{E}_\omega(u^c) = \limsup_{j\to+\infty} \mathcal{E}_{\omega_j}(u_j)$$

Taking the infimum over all c and using the definition of the functional $\mathcal{E}_\omega$, we obtain the desired inequality. □

Corollary 4.3.7. — *Let (X,D) be a log smooth log canonical pair (in particular, the coefficients of D are in $]-\infty,1]$) and assume that $L:=K_X+D$ is semi-positive and big. Fixing an ample line bundle A set $L_j:=L+A/j$. Let ψ_j be a decreasing sequence of smooth metrics on on the klt part D_{klt} of D such that $\psi_j\to\phi_{klt}$ (where $dd^c\phi_{klt}=[D_{klt}]$) and consider the Monge-Ampère equations*

$$(dd^c\phi_j)^n=e^{\phi_j-\psi_j-\phi_D}$$

for a metric $\phi_j\in\mathcal{E}(X,L_j)$. The equations admit unique solutions ϕ_j and moreover $\phi_j\to\phi_{KE}$ in L^1 where ϕ_{KE} is the unique solution of equation (4.3.4).

4.4. Smoothness of the Kähler-Einstein metric

Before we go into the details of the proof of the regularity theorem, we would like to give an overview of previous related results and underline the main differences that will appear in our specific case, namely the case of general log canonical pairs. As we will rely on the so called logarithmic case (ie (X,D) is log smooth, D is reduced, and K_X+D is ample), the next section will be devoted to recall some of the main tools appearing in this setting. Then, we will give the proof of the main regularity theorem, which will constitute the core of this section.

4.4.1. Special features in the log canonical case. — We should first mention the case of varieties with log terminal singularities, or more generally klt pairs, which correspond to the pairs where the discrepancies a_i defined earlier satisfy the strict inequalities $a_i>-1$. Then the situation is relatively well understood: In the non-positively curved case, we know that the Kähler-Einstein metric exists, is unique, has bounded potential, and induces on the regular locus a genuine Kähler-Einstein metric (see e.g. [**BEGZ10, EGZ08, EGZ09, DP10**]). As for the case of positive curvature, or log Fano manifolds, then there exist criteria (like the properness of the Mabuchi functional) to guarantee the existence and uniqueness (modulo automorphisms of X) of a Kähler-Einstein metric [**BBE+11**]; this metric is also known to have bounded potential and to be smooth on the regular locus of X (see again [**BBE+11, Pău08**]).

However, the behavior of the Kähler-Einstein metric near the singularities of X is mostly unknown (except if the singularities are orbifold). In the case of a klt pair, we know that the metric will not be smooth along the divisor, but its singularities can sometimes be understood outside of the singular part of (X,D). For example, a recent result in this direction states (under some technical assumption) that the Kähler-Einstein metric has cone singularities near each point where (X,D) is log smooth, ie X is smooth and D has simple normal crossing support (cf [**Gue12a**]).

When (X,D) is now a log smooth pair, the situation gets easier because there is no more loss of positivity coming from the resolution of singularities. For example, if the coefficients of D are in $[0,1)$ (the pair is then klt), the Kähler-Einstein metric is known to have cone singularities along D at least under some assumptions like D is smooth [**JMR11**], the coefficients are greater than $1/2$ [**CGP11**], or both [**Bre11**]. The general case has been announced in [**MR12**].

When now every coefficient of D is equal to 1, and $K_X + D$ is ample, then we know from the work of Kobayashi [**Kob84**] and Tian-Yau [**TY87**] that there exists a unique complete Kähler-Einstein metric having Poincaré singularities along D. The situation where the coefficients of D are in $[1/2, 1]$ behaves like a product of cone and Poincaré geometries and was studied in [**Gue12b**].

In a slightly different direction, Tsuji [**Tsu88a**] has considered the case of a singular variety with ample canonical line bundle such that only one divisor appears in its resolution, with discrepancy equal to -1. Finally, Wu [**Wu08, Wu09**] has worked out the case of a quasi-projective manifold compactified by a snc divisor $\sum D_i$ such that $K_X + \sum a_i D_i$ is ample for some coefficients $a_i \geqslant -1$. In our case however, such a strong positivity assumption will never happen as soon as we have to perform a non-trivial resolution.

As one can already observe in the log smooth case studied by Kobayashi and Tian-Yau, the log canonical case is very different from the klt case. Let us mention some striking divergences: first of all, the potentials are no more bounded even in the ample case so that the solution does not have minimal singularities. Moreover, the Kähler-Einstein equation in this setting is closely related to a negative curvature geometry. Indeed, if we first consider the Ricci-flat case, then it is impossible to write the equation on the whole X. Indeed, the current solution obtained on X_{reg} will not have finite mass near the singularities, and hence it will not extend as a global positive current on X. This phenomenon already happens in [**TY90**]. Finally, it has been proven in [**BBE**$^+$**11**, Proposition 3.8] that any pair (X, D) with X normal and $-(K_X + D)$ ample admitting a Kähler metric ω on X_{reg} with continuous potentials solution of $\mathrm{Ric}\,\omega = \omega + [D]$ is necessarily klt. Therefore it is pointless to look for positively curved Kähler-Einstein metric in the general setting of log canonical pairs instead of klt pairs.

To finish this discussion, let us stress the fact that the class of varieties with semi-log canonical singularities can be realized as a subclass of log canonical pairs (cf definition 4.2.8). This is the largest "reasonable" class to look for Kähler-Einstein metrics: for example, if X is a smooth Fano manifold carrying a smooth divisor $D \in |-K_X|$, then for any $\varepsilon > 0$, one has $K_X + (1+\varepsilon)D > 0$; however, there is no smooth Kähler-Einstein metric with negative scalar curvature on $X \setminus D$. Indeed, its existence would contradict the Yau-Schwarz lemma applied with the complete Ricci-flat Kähler metric constructed in [**TY90**].

Moreover, we will see that the existence of a negatively curved Kähler-Einstein metric on the regular part of a normal projective variety with maximal volume forces the singularities to be at worst log canonical, cf. Proposition 4.5.1.

4.4.2. The logarithmic case. — In this section, we will briefly recall the Theorem of Kobayashi and Tian-Yau constructing negatively-curved Kähler-Einstein metrics on quasi-projective varieties $X \setminus D$ where D is a reduced divisor with simple normal crossings, and $K_X + D$ is ample. In the course of the proof of Theorem 4.4.6, we will use in an essential manner the functional spaces introduced by these authors, namely the "quasi-coordinates" version of the usual Hölder spaces $\mathscr{C}^{k,\alpha}$. For now, X_0 will denote $X \backslash D$.

Definition 4.4.1. — We say that a Kähler metric ω on X_0 is of Carlson-Griffiths type if there exists a Kähler form ω_0 on X such that $\omega = \omega_0 - \sum_K dd^c \log\log \frac{1}{|s_k|^2}$.

In [**CG72**], Carlson and Griffiths introduced such a metric for some $\omega_0 \in c_1(K_X+D)$, but one can easily see that such a metric always exists on a Kähler manifold without assumptions on the bundle K_X+D. One can also observe that the existence of such a metric ω forces the cohomology class $\{\omega\}$ to be Kähler by Demailly's regularization theorem [**Dem82, Dem92**].

The reason why we exhibit this particular class of Kähler metrics on X_0 having Poincaré singularities along D is that we have an exact knowledge on its behaviour along D, which is much more precise that knowing its membership in the aforementioned class. This is precisely the class in which one will look for a Kähler-Einstein metric, so that one needs to define the appropriate analogue of the usual Hölder spaces $\mathscr{C}^{k,\alpha}$. And to do so, one may (almost) boil down to the usual euclidian situation.

The key point is that (X_0, ω) has bounded geometry at any order. Let us get a bit more into the details. To simplify the notations, we will assume that D is irreducible, so that locally near a point of D, X_0 is biholomorphic to $\mathbb{D}^* \times \mathbb{D}^{n-1}$, where $\mathbb{D}$ (resp. $\mathbb{D}^*$) is the unit disc (resp. punctured disc) of $\mathbb{C}$. We want to show that, roughly speaking, the components of ω in some appropriate coordinates have bounded derivatives at any order. The right way to formalize it consists in introducing quasi-coordinates: they are maps from an open subset $V \subset \mathbb{C}^n$ to X_0 having maximal rank everywhere. So they are just locally invertible, but these maps are not injective in general.
To construct such quasi-coordinates on X_0, we start from the univeral covering map $\pi : \mathbb{D} \to \mathbb{D}^*$, given by $\pi(w) = e^{\frac{w+1}{w-1}}$. Formally, it sends 1 to 0. The idea is to restrict π to some fixed ball $B(0,R)$ with $1/2 < R < 1$, and compose it (at the source) with a biholomorphism Φ_η of $\mathbb{D}$ sending 0 to η, where η is a real parameter which we will take close to 1. If one wants to write a formula, we set $\Phi_\eta(w) = \frac{w+\eta}{1+\eta w}$, so that the quasi-coordinate maps are given by $\Psi_\eta = \pi \circ \Phi_\eta \times \mathrm{Id}_{\mathbb{D}^{n-1}} : V = B(0,R) \times \mathbb{D}^{n-1} \to \mathbb{D}^*$, with $\Psi_\eta(v, v_2, \dots, v_n) = (e^{\frac{1+\eta}{1-\eta}\frac{v+1}{v-1}}, v_2, \dots, v_n)$.
Once we have said this, it is easy to see that X_0 is covered by the images $\Psi_\eta(V)$ when η goes to 1, and for all the trivializing charts for X, which are in finite number. Now, an easy computation shows that the derivatives of the components of ω with respect to the v_i's are bounded uniformly in η. This can be thought as a consequence of the fact that the Poincaré metric is invariant by any biholomorphism of the disc.

At this point, it is natural to introduce the Hölder space of $\mathscr{C}^{k,\alpha}_{qc}$-functions on X_0 using the previously introduced quasi-coordinates:

Definition 4.4.2. — For a non-negative integer k, a real number $\alpha \in]0.1[$, we define:

$$\mathscr{C}^{k,\alpha}_{qc}(X_0) = \{u \in \mathscr{C}^k(X_0);\ \sup_{V,\eta} ||u \circ \Psi_\eta||_{k,\alpha} < +\infty\}$$

where the supremum is taken over all our quasi-coordinate maps V (which cover X_0). Here $||\cdot||_{k,\alpha}$ denotes the standard $\mathscr{C}^{k,\alpha}_{qc}$-norm for functions defined on a open subset of $\mathbb{C}^n$.

The following fact, though easy, is very important (see e.g [**Kob84**] or [**Gue12b**, Lemma 1.6] for a detailed proof) :

Lemma 4.4.3. — *Let ω be a Carlson-Griffiths type metric on X_0, and ω_0 some Kähler metric on X. Then*

$$F_0 := \log\left(\prod |s_k|^2 \log^2 |s_k|^2 \cdot \omega^n/\omega_0^n\right)$$

belongs to the space $\mathscr{C}^{k,\alpha}_{qc}(X_0)$ for every k and α.

Finding the Kähler-Einstein metric consists then in showing that the Monge-Ampère equation $(\omega + dd^c\varphi)^n = e^{\varphi+f}\omega^n$ has a unique solution $\varphi \in \mathscr{C}^{k,\alpha}_{qc}(X_0)$ for all functions $f \in \mathscr{C}^{k,\alpha}_{qc}(X_0)$ with $k \geqslant 3$. This can be done using the continuity method in the quasi-coordinates. In particular, applying this to $f = F := -\log\left(\prod |s_k|^2 \log^2 |s_k|^2 \cdot \omega^n/\omega_0^n\right) + (\text{smooth terms on } X)$, which the previous lemma allows to do, this will prove the existence of a negatively curved Kähler-Einstein metric, which is equivalent to ω (in the *strong* sense: $\varphi \in \mathscr{C}^{k,\alpha}_{qc}(X_0)$ for all k, α).

In this continuity method, one needs to obtain first uniform estimates; they follow from a consequence of Yau's maximum principle for complete manifolds which we recall here (see [**CY80**, Proposition 4.1]):

Proposition 4.4.4. — *Let (X, ω) be a n-dimensional complete Kähler manifold, and $F \in \mathscr{C}^2(X)$ bounded from above. We assume that we are given $u \in \mathscr{C}^2(X)$ satisfying $\omega + dd^c u > 0$ and*

$$(\omega + dd^c u)^n = e^{u+F}\omega^n$$

Suppose that the bisectional curvature of (X, ω) is bounded below by some constant, and that u is bounded from below. Then

$$\inf_X u \geqslant -\sup_X F \quad \textit{and} \quad \sup_X u \leqslant -\inf_X F$$

There are similar results for the Laplacian estimates, but as we will not use them directly, we do not state them here. To summarize the discussion, one obtains:

Theorem 4.4.5 **(Kobayashi [Kob84], Tian-Yau [TY87]).** — *Let X be a compact Kähler manifold, D a reduced divisor with simple normal crossings, ω a Kähler form of Carlson-Griffiths type on $X\backslash D$, and $F \in \mathscr{C}^{k,\alpha}_{qc}(X\backslash D)$ for some $k \geqslant 3$. Then there exists $\varphi \in \mathscr{C}^{k,\alpha}_{qc}(X\backslash D)$ solution to the following equation:*

$$(\omega + dd^c\varphi)^n = e^{\varphi+F}\omega^n$$

In particular if $K_X + D$ is ample, then there exists a (unique) Kähler-Einstein metric of curvature -1 equivalent to ω.

4.4.3. Statement of the regularity theorem. — In this section, we prove that the Kähler-Einstein metric attached to a log canonical pair (X, D) (satisfying $K_X + D$ ample) by Theorem 4.3.2 is smooth on $X_0 = X_{\text{reg}} \setminus \text{Supp}(D)$. As usual, we will work on a log resolution $\pi : X' \to X$, where:

$$K_{X'} = \pi^*(K_X + D) + \sum a_i E_i$$

E_i being either an exceptional divisor or a component of the strict transform of D, and the coefficients a_i (called discrepencies) satisfy the inequality $a_i \geqslant -1$.

The Kähler-Einstein metric is given on X' by a (singular) psh weight ϕ on $\pi^*(K_X+D)$ satisfying

$$(dd^c\phi)^n = e^{\phi+\sum a_i\phi_{E_i}}$$

where ϕ_{E_i} is a psh weight on $\mathcal{O}_{X'}(E_i)$ such that $dd^c\phi_{E_i}=[E_i]$. So if in local coordinates, E_i is given by $\{z_n=0\}$, then $\phi_{E_i}=\log|z_n|^2$.

Our aim is to obtain regularity properties for the solutions of degenerate Monge-Ampère equations like the previous one; this is the content of the following theorem:

Theorem 4.4.6. — *Let X be a compact Kähler manifold of dimension n, dV some volume form, $D=\sum a_iD_i$ a $\mathbb{R}$-divisor with coefficients in $(-\infty,1]$ and defining sections s_i, $E=\sum c_\alpha E_\alpha$ an effective $\mathbb{R}$-divisor such that $D_{red}+E$ has snc support, and θ a semipositive form with $\int_X\theta^n>0$ such that $\{\theta\}-c_1(E)$ is a Kähler class. Then the θ-psh function φ with full Monge-Ampère mass, which is a solution of*

$$\langle(\theta+dd^c\phi)^n\rangle = \frac{e^{\varphi}dV}{\prod_i|s_i|^{2a_j}}$$

is smooth outside of $\mathrm{Supp}(D)\cup\mathrm{Supp}(E)$.

Note that although φ has full Monge-Ampère mass, it is in general far from having minimal singularities as soon as some coefficient a_i of D equals 1. Think for example of the logarithmic case (a log smooth pair (X,D) where K_X+D is ample; then the potential of the Kähler-Einstein metric is not bounded whereas the class is ample.

Let us go back to the general Kähler-Einstein case. We would like to apply the previous results with E being some positive combination of the E_i's. The problem is that there might be no such divisors; for example if π happens to be a small resolution, its exceptional locus has codimension at least 2. Therefore we need to perform another modification.

On X', $\pi^*(K_X+D)$ is no more ample, and by [**BBP10**, Proposition 1.5], its augmented base locus is $\mathbb{B}_+(\pi^*(K_X+D)) = \pi^{-1}(\mathbb{B}_+(K_X+D))\cup\mathrm{Exc}(\pi) = \mathrm{Exc}(\pi)$, and lies above $X_{\mathrm{sing}}\cup\mathrm{Supp}(D)$. It is well-known that one can find a log resolution $\mu: X''\to X'$ of $(X',\mathbb{B}_+(\pi^*(K_X+D)))$, and an effective $\mathbb{Q}$-divisor F with snc support lying above $\mathbb{B}_+(\pi^*(K_X+D))$ and such that $\mu^*\pi^*(K_X+D)-F$ is ample. Moreover one can also assume that $F+\sum E_i'$ has snc support, where E_i' denotes the strict transform of E_i by μ.

Let us recall the argument. We start by resolving the singularities of a Kähler current $T\geqslant\omega$ (ω a Kähler form on X') in $\pi^*(K_X+D)$ computing $\mathbb{B}_+(\pi^*(K_X+D))$, then we write Siu's decomposition $\mu^*T=\theta+[D]$ with θ semi-positive dominating $\mu^*\omega$, and D lying above $\mathbb{B}_+(\pi^*(K_X+D))$. Finally, we choose a μ-exceptional $\mathbb{Q}$-divisor G such that $-G$ is μ-ample; it exists because μ is a finite composition of blow-ups with *smooth* centers. Then it becomes clear that for $\varepsilon>0$ small enough, $\{\mu^*\theta\}-\varepsilon G$ is a Kähler class, and we have $\mu^*\pi^*(K_X+D)=(\{\mu^*\theta\}-\varepsilon G)+(\varepsilon G+D)$, with $\varepsilon G+D$ lying above $\mathbb{B}_+(\pi^*(K_X+D))$ and having simple normal crossing support. If one had chosen a log resolution of the ideal sheaf generated by the augmented base locus of $\pi^*(K_X+D)$ and the $\mathcal{O}'_X(E_i)$, we would have obtained the refined result that $F + E'$ has snc support.

Set $\nu := \pi \circ \mu : X'' \to X$, and write $K_{X''} = \nu^*(K_X + D) + E_\nu$. We know that E_ν is a divisor with snc support and coefficients $\geqslant -1$, and by the construction above, there exists a snc divisor F on X'' lying above $X_{\rm sing} \cup \mathrm{Supp}(D)$ such that $F + (E_\nu)_{\rm red}$ has snc support and $\nu^*(K_X + D) - F$ is ample. Applying Theorem 4.4.6, we get:

Corollary 4.4.7. — *Let (X, D) be a log canonical pair such that $K_X + D$ is ample. Then the Kähler-Einstein metric $\omega_{\rm KE}$ on (X, D) is smooth on $X_{\rm reg} \setminus \mathrm{Supp}(D)$.*

As we shall see in the course of the proof (cf §4.4.5.2), we do not obtain very precise estimates on the potential of the solution, even at order zero. However, it is tempting to believe that the potential $\phi_{\rm KE}$ of the Kähler-Einstein metric should be locally bounded outside of the non-klt locus of (X, D) defined as the support of the sheaf $\mathscr{O}_X/\mathscr{I}(X, D)$ where $\mathscr{I}(X, D)$ is the multiplier ideal sheaf of (X, D) (cf. e.g. [**Laz04**]). However, as this locus cannot be read easily on some log resolution, it does not seem obvious how one should tackle this question.

4.4.4. Preliminaries: the regularized equation. — We now borrow the notations of Theorem 4.4.6, and we let ω_0 be a Kähler form on X; it will be our reference metric in the following. Recall that we want to solve the equation

$$\mathrm{MA}(\varphi) = \frac{e^{\varphi} dV}{\prod_i |s_i|^{2a_i}}$$

where the unknown function is φ a θ-psh function, s_i are non-zero sections of $\mathscr{O}_X(D_i)$, $|\cdot|_i$ are smooth hermitian metrics on $\mathscr{O}_X(D_i)$, $f \in \mathscr{C}^\infty(X)$ and dV is a smooth volume form on X. Moreover, the expression $\mathrm{MA}(\varphi)$ has to be understood as the non-pluripolar Monge-ampère operator. It will be convenient for the following to differentiate the "klt part" of D from its "lc part", so we introduce the following notation:

$$D = \underbrace{\sum_{a_j<1} a_j D_j}_{D_{klt}} + \underbrace{\sum_{a_k=1} D_k}_{D_{lc}}$$

By Theorem 4.3.5, we know that the solution is the limit of any sequence of solutions of some appropriate regularized equations. The regularization process we are going to use concerns both the *a priori* non-Kähler class $\{\theta\}$ and the "klt part" in the volume form: $\prod_{a_j<1} |s_i|^{-2a_j}$. More concretely, we will be studying the following equation:

$$\langle(\theta + t\omega_0 + dd^c\varphi_{t,\varepsilon})^n\rangle = \frac{e^{\varphi_{t,\varepsilon}+f} dV}{\prod_{a_j<1}(|s_i|^2 + \varepsilon^2)^{a_i} \prod_{a_k=1} |s_k|^2} \tag{4.4.1}$$

Smoothness of the regularized solution. — At this point, it is still not completely clear that the solution $\varphi_{t,\varepsilon}$ of equation (4.4.1) is smooth on $X \setminus D_{lc}$. So we translate our equation into the logarithmic setting : we set

$$\omega_{t,lc} := \theta + t\omega_0 - \sum_{a_k=1} dd^c \log(\log |s_k|^2)^2$$

We may choose the hermitian metrics $|\cdot|_k$ such that $|s_k| < 1$ and such that $\omega_{t,lc}$ defines a Kähler metric on $X \setminus D_{lc}$ (cf [**CG72, Gri76**] e.g.). Of course this rescaling will depend on t, but we will explain how to bypass this problem later.

So using this new reference metric, one may rewrite equation (4.4.1) in the following form:

$$(\omega_{t,lc} + dd^c\psi_{t,\varepsilon})^n = \frac{e^{\psi_{t,\varepsilon}+f_t}\omega_{t,lc}^n}{\prod_{a_j<1}(|s_j|^2+\varepsilon^2)^{a_j}}$$

where $\psi_{t,\varepsilon} = \varphi_{t,\varepsilon} + \sum_{a_k=1}\log(\log|s_k|^2)^2$ and $f_t = -\log\left(\frac{\prod_k |s_k|^2\log^2|s_k|^2\omega_{t,lc}^n}{dV}\right)$. Clearly, f_t is bounded (but only the lower bound is uniform in t) and smooth on $X \setminus D_{lc}$, but we know by Lemma 4.4.3 that f_t is smooth when read in the quasi-coordinates adapted to the pair (X, D_{lc}). Therefore, using the Theorem of Kobayashi and Tian-Yau (see Theorem 4.4.5), we know that the solution $\psi_{t,\varepsilon}$ is bounded on $X \setminus D_{lc}$: there exists $C_{t,\varepsilon} > 0$ such that

$$-C_{t,\varepsilon} - \sum_{a_k=1}\log(\log|s_k|^2)^2 \leqslant \varphi_{t,\varepsilon} \leqslant C_{t,\varepsilon} - \sum_{a_k=1}\log(\log|s_k|^2)^2 \tag{4.4.2}$$

Moreover, $\psi_{t,\varepsilon}$ is smooth in the quasi-coordinates. In particular, $\omega_{t,lc} + dd^c\psi_{t,\varepsilon}$ is a Kähler metric with bounded geometry on $X \setminus D_{lc}$ and with Poincaré type growth along D_{lc}. Therefore it is complete and has a bounded curvature tensor. To prove the regularity theorem, we will thus have to obtain on each compact subset of X_0 estimates on the potential φ_ε at any order.

A first attempt at the uniform estimate. — The previous observation allows us to apply Yau's maximum principle (cf Proposition 4.4.4), and obtain that

$$\sup_{X\setminus D_{lc}}\psi_{t,\varepsilon} \leqslant \sup_{X\setminus D_{lc}}\left(\sum a_i\log(|s_i|^2+\varepsilon^2) - f_t\right)$$

and similarly $\inf_{X\setminus D_{lc}}\psi_{t,\varepsilon} \leqslant \inf_{X\setminus D_{lc}}\left(\sum a_i\log(|s_i|^2+\varepsilon^2) - f_t\right)$. If some coefficient a_i is negative, then we cannot obtain a bound for $\sup\psi_{t,\varepsilon}$. As for the lower bound, $-f_t$ is not uniformly bounded from below because $\omega_{t,lc}$ degenerates at $t=0$ (and if some a_i is positive, $a_i\log(|s_i|^2+\varepsilon^2)$ is not uniformly bounded below neither), so we cannot expect to find a lower bound for $\psi_{t,\varepsilon}$ using this strategy. Therefore we need another method to obtain a zero-order estimate on the potential of the solution. In fact, we will need to add some barrier function to gain positivity, in the spirit of Tsuji's trick [**Tsu88b**] for the Laplacian estimate of a degenerate class; the novelty in our situation is that this is also needed for the zero-order estimates (as opposed to the klt case).

4.4.5. Uniform estimate. — Before going any further, let us fix some notations.

4.4.5.1. A new framework. — We will denote by s_i, $i \in I$ (non-zero) sections of the (reduced) components of the divisor $D_{red} + E$, and by s_α, $\alpha \in A$ (non-zero) sections of the (reduced) components of E; we endow all these line bundles with suitable smooth hermitian metrics (se below). Finally, we set $X_0 := X \setminus (\mathrm{Supp}(D) \cup \mathrm{Supp}(E))$, and define $F := (D_{red} + E)_{red}$ as the reduced divisor $X \setminus X_0$.

The idea is to work on X_0. Of course, if we endow the last space with the Kähler metric $\omega_{t,lc}$, it will not be complete (near D_{klt} e.g.), so we won't be able to use Yau's maximum principle. Instead, we will rather use the following metric:

$$\omega_\chi := \theta + t\omega_0 - \sum_{i\in I} dd^c \log(\log|s_i|^2)^2 + dd^c\chi$$

where $\chi := \sum_\alpha c_\alpha \log|s_\alpha|^2$ (recall that the c_α's are the coefficients of E).

We do here a slight abuse of notation because ω_χ depends on t. However, the following lemma shows that the dependence is harmless:

Lemma 4.4.8. — *Up to changing the previously chosen hermitian metrics, the* $(1,1)$*-form* ω_χ *defines on* X_0 *a smooth Kähler metric with Poincaré growth along* F *having bounded geometry, all of those properties being satisfied uniformly in* t.

What we mean by this statement is that there exists a Poincaré-type metric ω_P on X_0 and a constant $C>0$ independent of t such that $C^{-1}\omega_P \leqslant \omega_\chi \leqslant C\omega_P$, and that in the appropriate quasi-coordinates attached to the pair (X,F), the coefficients $g_{i\bar{j}}$ of ω_χ satisfy $\left|\frac{\partial^{|\alpha|+|\beta|}g_{i\bar{j}}}{\partial z^\alpha \partial \bar{z}^\beta}\right| \leqslant C_{\alpha,\beta}$ for some constants $C_{\alpha,\beta}>0$ independant of t. In particular, ω_χ has a uniformly (in t) bounded curvature tensor.

Proof of Lemma 4.4.8. — We know that $\{\theta\} - c_1(E)$ is ample. Therefore, up to changing the hermitian metrics h_α on $\mathcal{O}_X(E_\alpha)$, we may suppose that $\eta := \theta - \sum c_\alpha \Theta_{h_\alpha}(E_\alpha)$ defines a smooth Kähler form on X (we designated by $\Theta_{h_\alpha}(E_\alpha)$ the curvature form of the hermitian line bundle (E_α, h_α)). Therefore, on X_0, we have:

$$\omega_\chi = \eta + t\omega_0 - \sum_{i\in I} dd^c \log(\log|s_i|^2)^2$$

and the statement follows easily from the computations of [**CG72**, Proposition 2.1] and [**Kob84**, Lemma 2] or [**TY87**]. □

4.4.5.2. Getting the lower bound. — First of all, we will use the crucial information that $\varphi_{t,\varepsilon}$ converges (in the weak sense of distributions) to some θ-psh function (cf first paragraph). By the elementary properties of psh (or quasi-psh) functions, we know that $(\varphi_{t,\varepsilon})$ is uniformly bounded above on the compact set X (see e.g. [**Hör94**, Theorem 3.2.13]). Therefore, we obtain some uniform constant C such that

$$\varphi_{t,\varepsilon} \leqslant C \tag{4.4.3}$$

Now, recall that we chose ω_χ to be the new reference metric, so our equation becomes

$$(\omega_\chi + dd^c u_{t,\varepsilon})^n = e^{u_{t,\varepsilon}+G_\varepsilon}\omega_\chi^n \tag{4.4.4}$$

where

$$u_{t,\varepsilon} := \varphi_{t,\varepsilon} + \sum_{i\in I}\log(\log|s_i|^2)^2 - \chi$$

and

$$G_\varepsilon := \chi + f + \sum_{a_j<1}\log\left(\frac{|s_j|^2}{(|s_j|^2+\varepsilon^2)^{a_j}}\right) - \log\left(\frac{\prod_{i\in I}|s_i|^2\log^2|s_i|^2\omega_\chi^n}{dV}\right)$$

Here again we should mention that G_ε also depends on t through the last term involving ω_χ. For our purpose, we can ignore this dependence in order to simplify the notations.

We can see from (4.4.3) and Lemma 4.4.8 that G_ε has a uniform (in t and ε) upper bound on X_0:

$$\sup_{X_0} G_\varepsilon \leqslant C$$

Moreover, we know from (4.4.2) that $\varphi_{t,\varepsilon} + \sum_{a_k=1} \log(\log |s_k|^2)^2$ is bounded. Therefore, it follows immediately that $u_{t,\varepsilon}$ is bounded *from below* (but a priori non uniformly). Applying Yau's maximum principle (cf. Proposition 4.4.4) to the smooth function $u_{t,\varepsilon}$ on the complete Kähler manifold (X_0, ω_χ) ensures that $\inf_{X_0} u_{t,\varepsilon} \geqslant -\sup_{X_0} G_\varepsilon \geqslant -C$. In terms of $\varphi_{t,\varepsilon}$, and recalling inequality (4.4.3) we get:

$$\boxed{C \geqslant \varphi_{t,\varepsilon} \geqslant \chi - C - \sum_{i\in I} \log(\log |s_i|^2)^2}$$

4.4.6. Laplacian estimate. — For the Laplacian estimate, we still work on the open manifold X_0. We endow it with the complete Kähler metric ω_χ, and we recall from Lemma 4.4.8 that ω_χ has uniformly bounded (bisectional) curvature.

As usual when one wants to compare to Kähler metrics ω and ω', the strategy is to use an inequality of the form $\Delta F \geqslant G$, where F, G involve terms like $\mathrm{tr}_\omega \omega'$, $\mathrm{tr}_{\omega'} \omega$ or the local potentials of $\omega' - \omega$. There exist several variants of such inequalities, due e.g. to Chern-Lu, Yau, Siu, etc. involving different assumptions on the curvature of the metrics involved. In our case, as we have a control on the bisectional curvature of the reference metric ω_χ, on the Ricci curvature of the "unknown metric" $\omega_\chi + dd^c u_{t,\varepsilon}$, and on the laplacian $\Delta_{\omega_\chi} G_\varepsilon$, we could use any of these formulas.

We have chosen to use a variant of Siu's inequality [**Siu87**, p.99], which can be found in [**CGP11**, Proposition 2.1] (see also [**Pău08, BBE**$^+$**11**]); notice the important feature allowing the factor e^{-F_-} for F_- quasi-psh which is crucial for us since the RHS of our Monge-Ampère equation has zeroes:

Proposition 4.4.9. — *Let X be a Kähler manifold of dimension n, ω, ω' two cohomologous Kähler metrics on X. We assume that $\omega' = \omega + dd^c u$ with $\omega'^n = e^{u+F_+-F_-}\omega^n$. and that we have a constant $C > 0$ satisfying:*

(i) $dd^c F_\pm \geqslant -C\omega$,

(ii) $\Theta_\omega(T_X) \geqslant -C\omega \otimes \mathrm{Id}_{T_X}$.

Then there exist some constant $A > 0$ depending only on n and C such that

$$\Delta_{\omega'}(\log \mathrm{tr}_\omega \omega' - Au + F_-) \geqslant \mathrm{tr}_{\omega'} \omega - nA.$$

Moreover, if one assumes that $\sup F_+ \leqslant C, u \geqslant -C$ and that $\log \mathrm{tr}_\omega \omega' - Au + F_-$ attains its maximum on X, then there exists $M > 0$ depending on n and C only such that:

$$\omega' \leqslant M e^{Au - F_-} \omega.$$

Here Δ (resp. Δ') is the laplacian with respect to ω (resp. ω'), and $\Theta_\omega(T_X)$ is the Chern curvature tensor of the hermitian holomorphic vector bundle (T_X, ω) (which may be identified with the tensor of holomorphic bisectional curvatures, usually denoted by the letter R).

Sketch of proof. — Siu's inequality applied to $\omega = \sum g_{i\bar{j}} dz_i \wedge d\bar{z}_k$ and $\omega' = \sum g'_{i\bar{j}} dz_i \wedge d\bar{z}_k$ yields:

$$\Delta'(\log \mathrm{tr}_\omega \omega') \geqslant \frac{1}{\mathrm{tr}_\omega \omega'}\big(- g^{j\bar{i}} R_{j\bar{i}} + \Delta(u + F_+ - F_-) + g'^{k\bar{l}} R^{j\bar{i}}_{k\bar{l}} g'_{j\bar{i}}\big)$$

Recollecting terms coming (with different signs) from the scalar and the Ricci curvature, we will obtain a similar inequality involving only a lower bound for the holomorphic bisectional curvature, namely

$$\Delta' \log \mathrm{tr}_\omega \omega' \geqslant \frac{\Delta(u + F_+ - F_-)}{\mathrm{tr}_\omega(\omega')} - B\, \mathrm{tr}_{\omega'} \omega \tag{4.4.5}$$

where B is a lower bound for the bisectional curvature of ω: this is the content of [**CGP11**, Lemma 2.2].
Clearly, $\Delta u = \mathrm{tr}_\omega \omega' - n$ so that $\Delta(u + F_+) \geqslant -n(C+1)$. As $\mathrm{tr}_\omega \omega' \mathrm{tr}_{\omega'} \omega \geqslant n$, we get

$$\frac{\Delta(u + F_+)}{\mathrm{tr}_\omega \omega'} \geqslant -(1+C)\mathrm{tr}_{\omega'} \omega \tag{4.4.6}$$

As for the second laplacian, we write

$$0 \leqslant C\omega + dd^c F_- \leqslant \mathrm{tr}_{\omega'}(C\omega + dd^c F_-)\omega'$$

and we take the trace with respect to ω:

$$\frac{nC + \Delta_\omega F_-}{\mathrm{tr}_\omega \omega'} \leqslant C \mathrm{tr}_{\omega'} \omega + \Delta_{\omega'} F_-$$

so that

$$\Delta_{\omega'} F_- \geqslant \frac{\Delta_\omega F_-}{\mathrm{tr}_\omega \omega'} - C \mathrm{tr}_{\omega'} \omega \tag{4.4.7}$$

Plugging (4.4.6) and (4.4.7) into (4.4.5), we get:

$$\Delta'(\log \mathrm{tr}_\omega \omega' + F_-) \geqslant -C_1 \mathrm{tr}_{\omega'} \omega$$

for $C_1 = 1 + B + 2C$. Finally, using $\Delta' u = n - \mathrm{tr}_{\omega'} \omega$, we see that

$$\Delta'(\log \mathrm{tr}_\omega \omega' - (C_1 + 1)u + F_-) \geqslant \mathrm{tr}_{\omega'} \omega - n(C_1 + 1)$$

which shows the first assertion by chosing $A := 1 + C_1$.

As for the second part, if we denote by p the point where the maximum is attained, then one has $(\mathrm{tr}_{\omega'} \omega)(p) \leqslant C_2$. Using the basic inequality $\mathrm{tr}_\omega \omega' \leqslant e^{u + F_+ - F_-} (\mathrm{tr}_{\omega'} \omega)^{n-1}$, one gets

$$\begin{aligned} \log(\mathrm{tr}_\omega \omega') &= (\log \mathrm{tr}_\omega \omega' - Au + F_-) + Au - F_- \\ &\leqslant (u(p) + F_+(p) - F_-(p)) + (n-1)\log(nA) - Au(p) + F_-(p) + Au - F_- \\ &\leqslant C_2 + Au - F_- \end{aligned}$$

where $C_2 = \sup F_+ + (n-1)\log(nA) - (A-1)\inf u$ (recall that A can be chosen to be positive). This concludes the proof of the proposition. □

Recall that we are interested in equation (4.4.4) given by

$$(\omega_\chi + dd^c u_{t,\varepsilon})^n = e^{u_{t,\varepsilon}+G_\varepsilon}\omega_\chi^n$$

We obtained the zero-order estimate on $u_{t,\varepsilon}$ in the last section, and now we want a Laplacian estimate. In order too use the previous proposition we first have to decompose G_ε as a difference of $C\omega_\chi$-psh functions in order to use the result above. Recall that

$$G_\varepsilon := \chi + f + \sum_{a_j<1} \log\left(\frac{|s_j|^2}{(|s_j|^2+\varepsilon^2)^{a_j}}\right) - \log\left(\frac{\prod_{i\in I}|s_i|^2\log^2|s_i|^2\omega_\chi^n}{dV}\right)$$

By [**Kob84**] or [**Gue12b**, Lemma 1.6], the last term is already known to be smooth in the quasi-coordinates (and it depends neither on t nor on ε).
We claim that

$$G_\varepsilon = \underbrace{\left[\chi + f + \sum_{a_j<1}\log|s_j|^2 + \sum_{a_j<0}\log(|s_j|^2+\varepsilon^2)^{-a_j}\right]}_{F_+} - \underbrace{\left[\sum_{0<a_j<1}\log(|s_j|^2+\varepsilon^2)^{a_j}\right]}_{F_-}$$

gives the desired decomposition $G_\varepsilon = F_+ - F_-$ in the notations of Proposition 4.4.9. Indeed, $\chi, f, \log|s_i|^2$ are quasi-psh, thus $C\omega_\chi$-psh for some uniform $C>0$ as ω_χ dominates some fixed Kähler form, cf Lemma 4.4.8. Moreover, a simple computation leads to the identity:

$$dd^c\log(|s|^2+\varepsilon^2) = \frac{\varepsilon^2}{(|s|^2+\varepsilon^2)^2}\cdot\langle D's, D's\rangle - \frac{|s|^2}{|s|^2+\varepsilon^2}\cdot\Theta_h$$

where Θ_h is the curvature of the hermitian metric h implicit in the term $|s|^2$, and $D's$ is the $(1,0)$-part of Ds where D is the Chern connection attached to $(\mathcal{O}_X(\mathrm{div}(s)), h)$. If follows that $F_\pm$ are $C\omega_\chi$-psh for some uniform $C>0$.

We can now apply Proposition 4.4.9 to the setting: $\omega = \omega_\chi, \omega' = \omega_\chi + dd^c u_{t,\varepsilon}, F_+ - F_- = G_\varepsilon$. Indeed, it is clear that F_+ is uniformly upper bounded, we just saw that $F_\pm$ are $C\omega_\chi$-psh, and we know from the previous section that $u_{t,\varepsilon}$ has a uniform lower bound. Furthermore, $\log\mathrm{tr}_\omega\omega' - Au + F_-$ attains its maximum on X_0: indeed, $-Au$ tends to $-\infty$ near the boundary of X_0, F_- is bounded (it is even smooth), and $\mathrm{tr}_\omega\omega' = \Delta_{\omega_\chi}(\varphi_{t,\varepsilon} + \sum_{i\in I}\log(\log|s_i|^2)^2 - \chi)$ is bounded on X_0 (we know it for the term $\Delta_{\omega_\chi}(\varphi_{t,\varepsilon}-\chi)$ and it is an elementary computation for the other term).

In conclusion, we may use Proposition 4.4.9 to obtain the following estimate:

$$\boxed{\theta + t\omega_0 + dd^c\varphi_{t,\varepsilon} \leqslant M\left(\prod_{i\in I}\log^2|s_i|^2\right)^C \cdot \prod_{\alpha\in A}|s_\alpha|^{-c_\alpha\cdot C}\prod_{a_j>0}|s_j|^{-2a_j}\,\omega_\chi}$$

For the "reverse inequality", we use the identity

$$(\omega_\chi + dd^c u_{t,\varepsilon})^n = e^{u_{t,\varepsilon}+G_\varepsilon}\omega_\chi^n$$

which leads to the inequality

$$\mathrm{tr}_{\omega_\chi + dd^c u_{t,\varepsilon}}\omega_\chi \leqslant e^{-(u_{t,\varepsilon}+G_\varepsilon)}\left(\mathrm{tr}_{\omega_\chi}(\omega_\chi + dd^c u_{t,\varepsilon})\right)^{n-1}$$

and therefore

$$\theta + t\omega_0 + dd^c\varphi_{t,\varepsilon} \geqslant M^{-1} \prod_{a_j<1} \frac{|s_j|^2}{(|s_j|^2+\varepsilon^2)^{a_j}} \cdot \left(\prod_{i\in I} \log^2 |s_i|^2\right)^{-C} \cdot \prod_{\alpha\in A} |s_\alpha|^{c_\alpha \cdot C} \prod_{a_j>0} |s_j|^{2a_j}\, \omega_\chi$$

for some uniform $C, M > 0$ (different from the previous ones).

In particular, for any compact set $K \Subset X_0$, there exists a constant $C_K > 0$ satisfying

$$C_K^{-1}\,\omega_0 \leqslant \theta + t\omega_0 + dd^c\varphi_{t,\varepsilon} \leqslant C_K\,\omega_0$$

Using Evans-Krylov theorem and the classical elliptic theory shows that the potential $\varphi_{t,\varepsilon}$ satisfies uniform $\mathscr{C}^{k,\alpha}$ estimates on any $\Omega \Subset K$ for each k, α. Thus the theorem is proved.

Remark 4.4.10. — One can easily obtain somewhat more precise estimates. Indeed, if α is a nef and big class and E an effective $\mathbb{R}$-divisor such that $\alpha - E$ is ample, then we have in fact that for every $\delta > 0$, $\alpha - \delta E$ is ample (write $\alpha - \delta E = (1-\delta)\alpha + \delta(\alpha - E)$). Applying this observation to Theorem 4.4.6, we see that for every $\delta > 0$, there exists $C_\delta > 0$ such that:

$$\varphi_{\mathrm{KE}} \geqslant \delta \sum c_\alpha \log |s_\alpha|^2 - \sum_{i\in I} \log(\log |s_i|^2)^2 - C_\delta$$

for every $\delta > 0$. One could apply the same argument to the Laplacian estimates.

4.4.7. Completeness. — Given a log canonical pair (X, D) such that $K_X + D$ is ample, we know that there exists a unique Kähler-Einstein current in $c_1(K_X + D)$ inducing a genuine Kähler-Einstein metric ω_{KE} on $X_0 = X_{\mathrm{reg}} \setminus \mathrm{Supp}(D)$ with maximal volume. In this section, we will try to understand whether the Kähler manifold $(X_0, \omega_{\mathrm{KE}})$ is complete. Let us first begin with the already known cases.

4.4.7.1. Previous results. — We choose a log resolution $\pi : X' \to X$ of (X, D) and write $K_{X'} = \pi^*(K_X + D) - E$, where $E = \sum a_i E_i$ is a divisor with snc support and coefficients $a_i \leqslant 1$.

Let us begin by the easiest case of a log smooth pair; ie $\pi = \mathrm{Id}_X$. Then, if $E = D$ is reduced, we known from [**Kob84, TY87**] that ω_{KE} has Poincaré singularities along E and hence is complete.

In fact, if (X, D) is a purely log canonical pair in the sense that for some resolution, E is reduced, then Yau showed in [**Yau93**] that the Kähler-Einstein metric is complete.

Coming back to the log smooth case, then if $E = \sum (1 - \frac{1}{m_i}) E_i$ has orbifold coefficients (ie $m_i \in \mathbb{Z}_{>0} \cup \{+\infty\}$), it follows from [**TY87**] that ω_{KE} is smooth in the orbifold sense along the strictly orbifold part of E (ie the E_i such that $m_i < +\infty$), and therefore ω_{KE} is complete if and only if E is reduced.

If now $E = \sum a_i E_i$ with $a_i \geqslant 1/2$ then it follows from [**Gue12b**] that ω_{KE} has mixed cone and Poincaré singularities; as cone metrics are never complete, one observes again that ω_{KE} is complete if and only if E is reduced.

Before we get to the main result of this section, let us give the two main tools we are going to use. The first one is the following consequence of Yau-Schwarz lemma in its version for volume forms [**Yau78a**, Theorem 3]:

Theorem 4.4.11. — *Let X be a complex manifold of dimension n endowed with two Kähler metrics ω_1 and ω_2. One assumes that the Ricci curvature of ω_1 is bounded from below by some constant $-K_1$, and that ω_2 has scalar curvature bounded from above by some negative constant K_2, and Ricci curvature bounded from below in a non-quantitative way. If ω_1 is complete, then we have:*

$$\omega_2^n \leqslant \frac{K_1}{K_2}\,\omega_1^n$$

The second tool is the following consequence of the fundamental results of [**BCHM10**] (cf e.g. [**EGZ09, BEGZ10, Gue12a**]) :

Theorem 4.4.12. — *If (X, D) is a klt pair of log general type (ie $K_X + D$ is big), then the Kähler-Einstein metric of (X, D) is smooth on $X_{\mathrm{reg}} \setminus (\mathrm{Supp}(D) \cup \mathbb{B}_+(K_X + D))$.*

4.4.7.2. Completeness of the Kähler-Einstein metric. — The goal of this section is to prove the following:

Theorem 4.4.13. — *Let (X, D) be a log canonical pair such that $K_X + D$ is ample, and let ω_{KE} be the smooth Kähler-Einstein metric on X_0. Then:*

(*i*) *If (X, D) is log smooth, then ω_{KE} is complete if and only if D is reduced.*
(*ii*) *If (X, D) is klt, then ω_{KE} is not complete, unless X is smooth and $D = 0$.*

In fact, we will see that point (*ii*) is valid for any klt pair (X, D) with $K_X + D$ big.

Remark 4.4.14. — In the positively curved case, Bonnet-Myers theorem shows that the Kähler-Einstein metrics are never complete (besides, the singularities are always klt by [**BBE$^+$11**, Proposition 3.8]).

Proof of Theorem 4.4.13. — Let us begin with the first point, and assume that ω_{KE} is complete for a log smooth pair (X, D). We write $D = D_{klt} + D_{lc}$ as usual, and we suppose that $D_{klt} \neq 0$. Let us choose some small open subset U of $X_{lc} = X \setminus D_{lc}$ where $D_{klt} = aH$ is given by one smooth hypersurface H with section s defined on X. Then we choose $\varepsilon > 0$ small enough, so that $K_X + D + \varepsilon H$ is still ample. Then there exists a Kähler-Einstein metric ω (with Ricci curvature equal to -1) for the pair $(X, D + \varepsilon H)$. We know from Lemma 4.4.15 that the local potentials φ_{KE} (resp. φ) of ω_{KE} (resp. ω) satisfy on U :

$$C_0 \geqslant \varphi_{\mathrm{KE}}, \quad \varphi \geqslant -\log\left(\log^2|s|^2\right) - C_0$$

for some constant $C_0 > 0$ depending only on U. Therefore, we have:

$$\omega^n \geqslant C_0 |s|^{-a-\varepsilon} \log^{-2}|s|^2$$

so that $\omega^n \geqslant C_1|s|^{-a-\varepsilon/2}$ for some $C_1 > 0$ depending only on U. By Yau-Schwarz lemma, we should have $\omega^n \leqslant \omega_{\mathrm{KE}}^n$. But as φ_{KE} is bounded from above on U, we also have $\omega_{\mathrm{KE}}^n \leqslant C_2|s|^{-a}$ on U, which is a contradiction.

Let us get to point (*ii*). We may assume that (X, D) is log smooth with $K_X + D$ semipositive and big. We choose a smooth hypersurface H which is a component of D. As for $\varepsilon > 0$ small enough, $(X, D + \varepsilon H)$ is klt and of log general type, we also get a Kähler-Einstein metric ω with -1 Ricci curvature smooth (at least) on the same locus as ω_{KE} (because $\mathbb{B}_+(K_X + D + \varepsilon H) \subset \mathbb{B}_+(K_X + D) \cup H$). Its potentials are known to have minimal singularities [**BEGZ10, Gue12a**] and in particular they have zero Lelong numbers. We can then get a contradiction exactly as in the previous case using Yau-Schwarz lemma. □

Let us now explain the proof of the following lemma that we used to show (*i*) in the previous theorem :

Lemma 4.4.15. — *Let (X, D) be a log smooth pair with $K_X + D$ ample. Then on any relatively compact subset $U \Subset X \setminus D_{lc}$, the local potentials φ_{KE} of the Kähler-Einstein metric satisfy:*

$$C \geqslant \varphi_{\mathrm{KE}} \geqslant -\sum_i \log\log^2(|s_i|^2) - C$$

where $C > 0$ depends only on U and the sum is taken over the section s_i of the components of D_{klt} restricted to U.

Proof. — This follows from the uniform estimate given in §4.4.5.2. Indeed, in the log smooth case, we do not need to use the barrier function χ because $K_X + D$ is already ample. □

We should mention that a simpler proof of the incompleteness of the Kähler-Einstein metric attached to a klt pair (X, D) with $K_X + D$ ample (or trivial) exists: indeed, in a log-resolution, this metric ω_{KE} is known to satisfy an estimate like $\omega_{\mathrm{KE}} \leqslant C_\varepsilon |s_E|^{-2\varepsilon} \prod_i |s_i|^{-2a_i} \omega$ where ω is smooth, and the relative canonical divisor of the resolution is $\sum a_i [s_i = 0]$, and E is an snc exceptional divisor, $\varepsilon > 0$ being arbitrary. Therefore the manifold $(X_0, \omega_{\mathrm{KE}})$ is easily seen to have finite diameter and hence cannot be complete unless X is smooth and $D = 0$. Note that this proof does not work neither in the big case, nor in the case (*i*) of Theorem 4.4.13.

Of course, Theorem 4.4.13 misses a lot of interesting cases. We conjecture that for a lc pair (X, D) with $K_X + D$ ample, the Kähler-Einstein metric ω_{KE} on X_0 is complete if and only if (X, D) has purely log canonical singularities in the sense that the klt locus should coincide with $X_0 = X_{\mathrm{reg}} \setminus \mathrm{Supp}(D)$; where we recall that $\mathrm{KLT}(X, D) := X \setminus \mathrm{Supp}(\mathcal{O}_X / \mathscr{I}(X, D))$, $\mathscr{I}(X, D)$ being the multiplier ideal attached to (X, D), cf [**Laz04**].

About uniqueness of the Kähler-Einstein metrics. — First of all, in the case where X is smooth and K_X is ample, then uniqueness of the Kähler-Einstein metric constructed by Aubin and Yau is a straightforward consequence of the maximum principle. Generalizing this principle to some complete Kähler manifolds in [**Yau75**], Yau could prove that on a Kähler manifold, there can be only one *complete* Kähler metric ω satisfying $\mathrm{Ric}\, \omega = -\omega$. In particular, this result has been applied by Kobayashi and Tian-Yau to show the uniqueness of the Kähler-Einstein metric for a log smooth pair (X, D) satisfying $K_X + D$ ample, cf [**Kob84, TY87**]. Using Yau-Schwarz lemma in a more subtle way (through the notion of almost complete metric), they also show uniqueness when $K_X + D$ is only assumed nef, big and ample modulo D, which means that $K_X + D$ intersects positively every curve not contained in D. For example, if (X, D) is a log resolution of some canonically polarized singular variety, these assumptions are not satisfied.

In our situation we proceed in a different manner: we first use the volume assumption (as a replacement for completeness) to show that the Kähler-Einstein metric, originally defined on the (log) regular locus , extends to define a a positive current on X whose local potentials glue to define a solution with full Monge-Amère mass of a global Monge-Ampère equation to which we can apply the comparison principle to finally deduce the uniqueness.

4.5. Applications

4.5.1. Yau-Tian-Donaldson conjecture for singular varieties. — Let us start with the following converse of Theorem A stated in the introduction.

Proposition 4.5.1. — *Let X be projective variety satisfying the conditions G_1 and S_2, and such that K_X is $\mathbb{Q}$-ample. If X admits a Kähler-Einstein metric ω in the following sense: ω is a K-E metric on X_{reg} and its total volume there is equal to K_X^n, then X has semi-log canonical singularities.*

Proof. — We begin with the case where X is normal.

Let us first show that $\phi := \log \omega^n$ on X_{reg} extends to an element in $\mathcal{E}(X, K_X)$. By the K-E equation $dd^c\phi := -\text{Ric}\ \omega = \omega \geqslant 0$ and hence, since X is normal, ϕ extends to a positively curved singular metric on K_X over all of X. Thus, writing $\omega = dd^c\psi$ for some other such metric ψ the compactness of X forces the K-E equation $\text{MA}(\phi) = e^{\phi}$ (up to shifting ϕ by a constant) globally on X (since the non-pluripolar MA does not charge the singular locus of X). Moreover, by the volume assumption we have that $\int_X \text{MA}(\phi) = K_X^n$ and hence $\phi \in \mathcal{E}(X, K_X)$, as desired.

Next, fix a resolution $\pi : X' \to X$ and assume, to a get a contradiction, that $p^*K_X = K_{X'} + D$ where D is a snc $\mathbb{Q}-$divisor such that $D = D' + (1+\delta)E$ for some $\delta > 0$, where D' is a $\mathbb{Q}-$divisor and E is a smooth irreducible divisor transversal to the support of D'. Since ϕ has maximal MA-mass it follows, as shown in [**BBE+11**] using an Izumi type estimate, that $\pi^*\phi$ has no Lelong numbers. In particular, it follows from the characterization of Lelong numbers that there exists a neigbourhood U of E such that $\pi^*\phi \geqslant \frac{1}{2}\delta \log |s_E|^2 - C$ in local trivializations. Moreover, we may take U such that D does not intersect U. But then it follows from the K-E equation that

$$K_X^n \geqslant C' \int_U e^{\pi^*\phi - (1+\delta)\log|s_E|^2} \geqslant C'' \int_U e^{-(1+\delta/2)\log|s_E|^2} = \infty,$$

which gives the desired contradiction.

We move on to the general case when X is only assumed to be G_1 and S_2. As we observed in §4.2.3, the result of Proposition 4.2.6 holds actually in the general G_1 and S_2 case (we did not use at all that the singularities were slc); however we should be careful and work instead on a log-resolution of (X^ν, C_{X^ν}) because the formula $\nu^*K_X = K_{X^\nu} + C_{X^\nu}$ could not be meaningful anymore if C_{X^ν} is not Cartier, cf §4.2.3 and the remarks following the identity (4.2.1). So the first conclusion is that the weight $\phi := \log \omega^n$ on X_{reg} extends on the normalization X^ν to a psh weight in $\mathcal{E}(X^\nu, \nu^* K_X)$. Then we take a log-resolution $\pi : X' \to X^\nu$ of the pair (X^ν, C_{X^ν}) where C_{X^ν} is the conductor of the normalization; a priori, this is just an effective divisor, possibly non-reduced. We write (X', D') for the new pair that we obtain on X'. Then same arguments as earlier show that D' has coefficients less than or equal to 1, which amounts to saying that (X^ν, C_{X^ν}) is log canonical, or equivalently that X has semi-log canonical singularities. □

To relate this to K-stability we recall that Odaka [**Oda08**] has shown that, if X is K-semistable, then X has semi-log canonical singularities (recall that we assume that X is G_1 and S_2 and that $K_X > 0$). Conversely, if X is semi-log canonical, then X is K-stable [**Oda11**]. Hence, combining our results with Odaka's results gives the following confirmation of the Yau-Tian-Donaldson conjecture for varieties being G_1 and S_2, with K_X ample:

Theorem 4.5.2. — *Let X be a G_1 and S_2 projective variety such that K_X is ample. Then X admits a Kähler-Einstein metric iff X is K-stable.*

It would be interesting to have a direct analytical proof of the implication "Kähler-Einstein implies K-stable" as shown in [**Ber12**] the (log) Fano case (where K-stability has to be replaced by K-polystability in the presence of holomorphic vector fields).

4.5.2. Automorphism groups of canonically polarized varieties. — The existence and uniqueness of Kähler-Einstein metrics established in Theorem A allows us to give an analytical proof of the following result shown in [**BHPS12**] (where two proofs were given, one cohomological and one geometric)

Proposition 4.5.3. — *Let X be a normal stable variety. Then X admits no non-trivial infinitisimal automorphisms.*

Proof. — By general results on automorphism groups of normal varieties (see [**BBE$^+$11**, Lemma 5.2] and references therein) it is equivalent to show that any holomorphic vector field V on X_{reg} vanishes identically. To prove this vanishing we first observe that, by normality, V is the infinitisimal generator of a complex one-parameter family of automorphism F of X and in particular of X_{reg}. Fix a Kähler-Einstein current ω on X. By the naturality of the KE-equation it follows that $F^*\omega$ is also a KE-current and hence by uniqueness $F^*\omega = \omega$ on X_{reg}. Let us denote by V_r and V_i the real and imagnary parts of V, which are infinitisimal generators of real one parameter families of automorphisms that we will denote by F_r and F_i respectively, which, by the previous argument, also preserve ω. Next, note that any automorphism automatically lifts to the line bundle K_X over X_{reg} and thus it follows from general principles that the real part V_r of V is a Hamiltonian vector field, i.e. $i_{V_r}\omega = dh$ for some smooth function h on X_{reg}. But then, by Cartan's formula, the Lie deriviative $L_{V_i}\omega$ is given by $d(i_{W_r}\omega) = dJi_{V_r}\omega = dJdh = dd^c h$. Since the flow F_i defined by V_i also preserves ω it thus follows that $dd^c h = 0$. But by normality it follows that $h = 0$ (indeed, by normality h is bounded and we can thus apply the maximum principle on a resolution). Hence $i_{V_r}\omega = 0$ on X_{reg}, which forces $V_r = 0$ on X_{reg}, since ω is Kähler there and in particular pointwise non-degenerate. Finally, by the same argument $V_i = 0$ (for example replacing V with JV) and hence $V = 0$ as desired. □

In the case when X is smooth there is a simple cohomological proof of the previous proposition: by Serre duality $H^0(X, TX)$ is isomorphic to $H^{n-1}(X, -K_X)$, which is trivial by Kodaira vanishing (since K_X is ample). In the case when X is log canonical a similar cohomological argument can be used [**BHPS12**], relying on the Bogomolov-Sommese vanishing result for log canonical singularities, established in [**GKKP11**, Theorem 7.2]. Indeed, if V does not vanish identically then contracting with V on X_{reg} maps K_X to a rank one reflexive sheaf in Hom $(K_X, \Omega_X^{[n-1]})$, where $\Omega_X^{[n-1]}$ is the sheaf of reflexive $(n-1)-$forms on X and hence by the Bogomolov-Sommese vanishing result in [**GKKP11**] the Kodaira dimension of K_X is at most $n-1$, which contradict the ampleness of K_X.

4.6. Outlook

4.6.1. Towards Miyaoka-Yau type inequalities. — For simplicity we will only consider the case $n = 2$ (but a similar discussion applies in the general case). We set $E := \Omega_X^1$, the cotangent bundle of X. The classical case is when X is smooth with K_X ample, where the Miyaoka-Yau inequality says

$$c_1(E)^2 \leqslant 3c_2(E).$$

Let us briefly recall Yau's differential-geometric proof. We equip E with the Hermitian metric induced by ω and denote by (E, ω) the corresponding Hermitian vector bundle. Then, if ω is Kähler-Einstein a direct local calculation gives the *point-wise* inequality

$$c_1(E, \omega)^2 \leqslant 3c_2(E, \omega)$$

formulated in terms of the Chern-Weil representatives $c_i(E, \omega)$ of the corresponding Chern classes. Hence, integrating immediately gives the Miyaoka-Yau inequality. Repeating this argument in the singular case when X a stable surface and using Theorem A gives the following

Proposition 4.6.1. — *The following inequality holds for a stable surface equipped with the canonical Kähler-Einstein metric ω on its regular part:*

$$c_1(K_X)^2 \leqslant 3 \int_{X_{\text{reg}}} c_2(E, \omega)$$

with equality iff ω has constant holomorphic sectional curvature, i.e. (X_{reg}, ω) is locally isometric to a ball.

Proof. — Since the point-wise inequality above still holds, by the KE-condition, we can simply integrate it over X_{reg} and use that, by Theorem A, $c_1(K_X)^2 = \int_{X_{\text{reg}}} c_1(E, \omega)^2$. The conditions for equality are well-known in the point-wise inequality. □

Since ω is canonically attached to X one could simply define the rhs appearing in the inequality above as the "analytical second Chern number" $c_{2,\text{an}}(X)$ of X. However, it should be stressed that it is not even a priori clear that $c_{2,an}(X)$ is finite, even though we expect that this is the case. More precisely, we expect that $c_{2,\text{an}}(X)$ can be identified with (or at least bounded from above) by a suitable algebraically defined second Chern class number $c_2(X)$. Various definitions of such Chern numbers have been proposed in the litterature and we refer the reader to the paper of Langer [**Lan00**] where very general algebraic Miyaoka-Yau type inequalities are obtained, which in particular apply to stable surfaces. More generally, as before, our arguments apply to log canonical pairs.

4.6.2. The Weil-Petersson geometry of the moduli space of stable varieties. — In this section we will briefly explain how the finite energy property of the Kähler-Einstien metric on a stable variety naturally appears in the geometric study of the moduli space $\mathcal{M}$ of all stable varieties. In a nut shell, the Kähler-Einstein metrics on stable varieties induces a metric on the $\mathbb{Q}-$line bundle $\mathcal{L} \to \mathcal{M}$ over the moduli space defined by the top Deligne pairing of K_X and the finite energy condition is precisely the condition which makes sure that the metric is point-wise finite. The relation to Weil-Petterson geometry comes from the well-known fact that the curvature form of the corresponding metric over the moduli space $\mathcal{M}_0$ of all *smooth* stable varieties (i.e. all canonically polarized $n-$dimensional manifolds, with a fixed oriented smooth structure) coincides with the Weil-Petersson metric Ω_{WP} on $\mathcal{M}_0$ [**FS90, Sch12**].

To be a bit more precise we first recall that given a line bundle $L \to X$ over a (complex) $n-$dimensional algebraic variety X its top Deligne pairing i.e. the $(n+1)-$fold Deligne pairing of L with itself is a complex line that we will denote by $\langle L\rangle$ [**Elk89, Elk90**]. Equipping $\langle L\rangle$ with an Hermitian metric ϕ (using additive notation as before) induces a Hermitian metric $\langle\phi\rangle$ on $\langle L\rangle$, satisfying the change of metric formula: $\langle\phi\rangle - \langle\psi\rangle = (\mathcal{E}(\phi) - \mathcal{E}(\psi))$ (up to a multiplicative normalization constant), where $\mathcal{E}$ is the energy functional appearing in §4.3 (compare [**PS04**]). Fixing a smooth reference metric ϕ_0 on L one can use the latter transformation formula to define the metric $\langle\phi\rangle$as long as ϕ has finite energy. The resulting metric $\langle\phi\rangle$ is then independent of the choice of reference metric ϕ_0. More generally, in the relative case of a flat morphism $\mathcal{X} \to B$ between integral schemes of relative dimension n and an Hermitian line bundle $L \to \mathcal{X}$ this construction produces an Hermitian line bundle $\langle L\rangle \to B$ over the base B.

In particular, taking X to be an $n-$dimensional stable variety $L := K_X$ one obtains a canonical metric on the complex line $\langle K_X\rangle$, induced by the finite energy metric on K_X determined by the volume form of the Kähler-Einstein metric on the regular locus of X.

Let now $\mathcal{M}$ denote the moduli space of all n-dimensional stable varieties with a fixed Hilbert polynomial [**Kol10, Kol**]. Using the existence of a universal stable family $\mathcal{X}$ (in the sense of Kollar) over a finite cover of each irreducible component of the moduli space one obtains a $\mathbb{Q}-$ line bundle $\mathcal{L}$ over $\mathcal{M}$, induced by the fiber-wise top Deligne pairings $\langle K_X\rangle$. We conjecture that the metric on $\mathcal{L}$ induced by the fiber-wise Kähler-Einstein metrics is continuous (as in the case of stable curves [**Fre12**]). Confirming this conjecture would require a more detailed analysis of the dependence of the Kähler-Einstein metric on the complex structure that we leave for the future.

As is well-known the curvature of the corresponding metric over $\mathcal{M}_0$ coincides (up to a numerical factor) with the Weil-Petersson metric Ω_{WP}. In particular, it is stricly positive as a form (in the orbifold sense). Under the validity of the previous conjecture one thus obtains a canonical extension of the induced Weil-Petersson metric on $\mathcal{M}_0$ to its compactification in $\mathcal{M}$ as a positive current with continuous potentials. It would also be very interesting to know under which assumptions the extension is *strictly* positive in a suitable sense, for example if it is, locally, the restriction of a Kähler metric? These problems are (e.g. by Grauert's generalization of Kodaira's embedding theorem to singular varities) closely related to the problem of showing that the correponding line bundle $\mathcal{L}$ over the moduli space $\mathcal{M}$ is ample (on each irreducible component) and it should be compared with the recent work of Schumacher [**Sch12**], where an analytic proof of the quasi-projectivity of $\mathcal{M}_0$ is given. As shown by Schumacher the Weil-Petersson metric Ω_{WP} on $\mathcal{M}_0$ admits a (non-canonical) extension as a positive current with analytic singularities to Artin's Moishezon compactifaction of $\mathcal{M}_0$. But the conjecture above is closely related to the problem of obtaining a *canonical* extension of Ω_{WP} to $\mathcal{M}$ as a positive current with continuous potentials.

Finally, it should be pointed out that the top Deligne pairing used above essentially coincides with Tian's CM-line bundle in this setting (by the Knudson-Mumford expansion and Zhang's isomorphism realizing the Chow divisor as a top Deligne pairing). The ampleness of the induced CM-line bundle over general moduli spaces of K-stable polarized varieties was recently speculated on by Odaka [**Oda13**].

CHAPITRE 5

SEMI-STABILITY OF THE TANGENT SHEAF OF SINGULAR VARIETIES

Dans ce dernier chapitre, nous changeons un peu de perspective, au sens où nous allons utiliser notre connaissance des métriques de Kähler-Einstein singulières afin de démontrer une généralisation d'un résultat bien classique de géométrie algébrique dans un cadre singulier. Plus précisément étant donnée une variété X à singularités log canoniques et telle que K_X soit ample, nous montrons que son faisceau tangent $\mathscr{T}_X$ est polystable par rapport à K_X. Toujours pour une variété X à singularités log canoniques, mais dans le cas où K_X est numériquement trivial, nous montrons la semi-stabilité de $\mathscr{T}_X$ par rapport à toute polarisation; et enfin, nous prouvons la semi-positivité générique du faisceau cotangent (resp. tangent) dans le cas où K_X (resp. $-K_X$) est nef.

Ces résultats illustrent bien l'idée générale selon laquelle l'étude des métriques KE singulières permet dans une certaine mesure de mieux comprendre la géométrie de variétés singulières de même que cela a été le cas avec les métriques KE habituelles, où les théorèmes d'Aubin et de Yau ont eu de nombreuses conséquences dans l'étude de la géométrie (algébrique) des variétés lisses.

Introduction

In this chapter, we show that the tangent sheaf of a canonically polarized variety X having log canonical singularity is polystable with respect to K_X. The semi-stability property is proved using the same line of arguments as Enoki [**Eno88**] for the case of canonical singularities : the strategy is to use approximate Kähler-Einstein metric on a resolution X', and consider them as approximate Hermite-Einstein metrics on $T_{X'}$. Then, one has to control the error terms in the semi-stability inequality which require certain estimates for singular Kähler-Einstein metrics. Finally, to prove the polystability, we have to adapt carefuly the usual arguments in the singular setting but in the end we really need the new input of [**BG13**] to conclude.

Adapting the proof, we can also get the semi-stability of the tangent sheaf of log canonical compact Kähler spaces with numerically trivial canonical bundle, and obtain the generic semipositivity of the cotangent (resp. tangent) sheaf of log canonical compact Kähler spaces with nef canonical (resp. anticanonical) bundle, in the spirit of Miyaoka generic semipositivity theorem.

5.1. Generalities

In this section, we briefly recall the definitions of slope, semi-stability before exposing the previously known results. We refer to [**Har80**], [**Kob87**, Chap. V] or [**HL10**] for more details.
In the following, X will be a complex projective normal variety of dimension n, and $\mathscr{F}$ will always denote a coherent sheaf. We write $\mathscr{F}^* = \mathscr{H}om(\mathscr{F}, \mathcal{O}_X)$ for the dual of $\mathscr{F}$. We say that $\mathscr{F}$ is reflexive if the natural map

$$j : \mathscr{F} \to \mathscr{F}^{**}$$

is an isomorphism. For instance, the dual of a coherent sheaf is always reflexive. Moreover, as the kernel of j is exactly the torsion of $\mathscr{F}$, a reflexive sheaf is automatically torsion-free.
We define the rank of a coherent sheaf $\mathscr{F}$ to be its rank at the generic point (or equivalently, consider the Zariski open subset where $\mathscr{F}$ is locally free), we denote if by $\operatorname{rk}\mathscr{F}$.

We need now to define the determinant of a coherent sheaf $\mathscr{F}$. We set $r = \operatorname{rk}\mathscr{F}$, and we let

$$\det \mathscr{F} := (\Lambda^r \mathscr{F})^{**}$$

be the determinant of $\mathscr{F}$. It is a rank one reflexive sheaf on X, but if X is not smooth, it is in general not a line bundle, ie it is not locally free. As X is normal, there is a $1-1$ correspondence between rank one reflexive sheaves (up to isomorphism) and Weil divisors (up to linear equivalence), the correspondence being given in the usual way $D \mapsto \mathcal{O}_X(D) = \{f, \operatorname{div}(f) \geqslant -D\}$ in one direction, and in the other direction, given $\mathscr{F}$ a rank one reflexive sheaf, we choose a Weil divisor on X_{reg} representing the line bundle $\mathscr{F}_{|X_{\text{reg}}}$ and take its closure.
We will denote by $c_1(\mathscr{F})$ the equivalence class of any Weil divisor attached to $\det \mathscr{F}$. We can now define the slope:

Definition 5.1.1. — Let A be an ample line bundle on X. We define the slope of $\mathscr{F}$ with respect to A to be the rational number

$$\mu_A(\mathscr{F}) := \frac{c_1(\mathscr{F}) \cdot A^{n-1}}{\operatorname{rk}\mathscr{F}}$$

Let us get now to the definition of slope stability, which goes back to Mumford [**Mum63**] and Takemoto [**Tak72**]:

Definition 5.1.2. — Let $\mathscr{E}$ be a torsion-free coherent sheaf on X, and A be an ample line bundle.
We say that $\mathscr{E}$ is *semi-stable* (resp. *stable*) with respect to A if for every coherent subsheaf $\mathscr{F} \subset \mathscr{E}$ (resp. every non-zero and proper coherent subsheaf $\mathscr{F}$), we have

$$\mu_A(\mathscr{F}) \leqslant \mu_A(\mathscr{E}) \quad (\text{resp.}\, \mu_A(\mathscr{F}) < \mu_A(\mathscr{E}))$$

We say that $\mathscr{E}$ is *polystable* (with respect to A) if $\mathscr{E}$ is the direct sum of stable subsheaves with same slope.

The Kobayashi-Hitchin correspondence states that on a compact complex manifold, a vector bundle E is polystable if and only if it admits a Hermite-Einstein metric [**Kob82,**

Lüb83, Don85, UY86]. As a consequence of the "easy" direction Hermite-Einstein implies polystable, whenever X admits a Kähler-Einstein metric ω_{KE}, then T_X is polystable with respect to $\{\omega_{\mathrm{KE}}\}$. In particular, when K_X is ample, it follows from the theorem of Aubin and Yau [**Aub78, Yau78b**] that T_X is K_X-polystable.

Using this idea, Enoki [**Eno88**] managed to prove that the tangent sheaf of a canonically polarized normal variety with canonical singularities is semi-stable with respect to K_X. The strategy of the proof is to work on a resolution, and build there smooth approximate Kähler-Einstein metrics. They induce approximate Hermite-Einstein metrics on T_X (in a naive sense), and using the fact that the singularities are canonical, he manages to control the error terms or at least the one not coming with a good sign in view of the semi-stability property.

In this paper, we follow this approach, but our main contribution (outside of the *polystability*) is to deal with the bad error terms coming from the log canonical singularities.

5.2. Chern-Lu formula

This section is devoted to the technical Proposition 5.2.1 which can be seen as a generalization of the Laplacian estimate obtained using Chern-Lu formula. It will be used in the next section to control the error term in the semi-stability inequality.

So let us first recall Chern-Lu's formula [**Che68, Lu68**], which is going to be a essential tool to get the laplacian estimate. Let (X, ω_X) and (Y, ω_Y) be two Kähler manifolds, and $f : X \to Y$ an holomorphic map satisfying $\partial f \neq 0$. Then

$$\Delta_{\omega_X} \log |\partial f|^2 \geqslant \frac{\mathrm{Ric}\,\omega_X \otimes \omega_Y(\partial f, \overline{\partial f})}{|\partial f|^2} - \frac{\omega_X \otimes R^Y(\partial f, \overline{\partial f}, \partial f, \overline{\partial f})}{|\partial f|^4}$$

where ∂f is viewed as a section of $T_X^* \otimes T_Y$.

Using this formula when f is the identity map (but the Kähler forms differ), one can derive so-called laplacian estimates for the Kähler-Einstein equation provided that the reference metric has upper bounded holomorphic bisectional curvature (cf [**JMR11**, Section 7]). The following proposition, inspired by [**Pău08**] (see also [**BBE$^+$11**, Theorem 10.1]), enables to derive (weaker) laplacian estimates in some cases where the Ricci curvature is not bounded from below:

Proposition 5.2.1. — *Let X be a compact Kähler manifold of dimension n, ω, ω' two cohomologous Kähler metrics on X. We assume that $\omega' = \omega + dd^c\varphi$ with $\omega'^n = e^{\psi^+ - \psi^-}\omega^n$ for some smooth functions $\psi^\pm$, and that we have a constant $C > 0$ satisfying:*

(*i*) $\sup_X |\varphi| \leqslant C$,
(*ii*) $\sup_X \psi^+ \leqslant C$ and $dd^c\psi^\pm \geqslant -C\omega$,
(*iii*) $\Theta_\omega(T_X) \leqslant C\omega \otimes \mathrm{Id}_{T_X}$.

Then there exists some constant $M > 0$ depending only on n and C such that

$$\omega' \geqslant M^{-1} e^{\psi^+} \omega.$$

Proof. — The main difficulty is that we do not have a control on the lower bound of $\mathrm{Ric}\,\omega' = \mathrm{Ric}\,\omega + dd^c\psi^- - dd^c\psi^+$. The trick, inspired by [**Pău08**], is to add ψ^+ in the laplacian appearing in Chern-Lu formula. Let us now get into the details. We apply Chern-Lu's formula to $f = \mathrm{id} : (X, \omega') \to (X, \omega)$. Then $|\partial f|^2 = \mathrm{tr}_{\omega'}\omega$. We denote by g (resp. h) the hermitian metrics induced by ω' (resp. ω). The second term of Chern-Lu formula is easily dealt with:

$$\begin{aligned} \omega' \otimes R^h(\partial f, \overline{\partial f}, \partial f, \overline{\partial f}) &= -g^{i\bar{j}} g^{k\bar{l}} R^h_{i\bar{j}k\bar{l}} \\ &\geqslant -C g^{i\bar{j}} g^{k\bar{l}} (h_{i\bar{j}} h_{k\bar{l}} + h_{i\bar{l}} h_{k\bar{j}}) \\ &\geqslant -2C(\mathrm{tr}_{\omega'}\omega)^2 \end{aligned}$$

Now recall that if α and β are two hermitian $(1,1)$ forms, and if $(\alpha, \beta)_{\omega'}$ denotes the hermitian product induced by ω', then we have

$$(\alpha, \beta)_{\omega'} = \mathrm{tr}_{\omega'}\alpha \cdot \mathrm{tr}_{\omega'}\beta - n(n-1) \frac{\alpha \wedge \beta \wedge \omega'^{n-2}}{\omega'^n}$$

Moreover, one can check that

$$\alpha \otimes \beta(\partial f, \overline{\partial f}) = (\alpha, \beta)_{\omega'}$$

(in the tensor product, α stands for the hermitian form induced by α on T_X^* relatively to ω'). As $\mathrm{Ric}\,\omega' \geqslant -nC\omega - (C\omega + dd^c\psi^+)$, using the two previous identities, we get:

$$\mathrm{Ric}\,\omega' \otimes \omega(\partial f, \overline{\partial f}) \geqslant -nC(\mathrm{tr}_{\omega'}\omega)^2 - \mathrm{tr}_{\omega'}\omega \cdot \mathrm{tr}_{\omega'}(C\omega + dd^c\psi^+)$$

At that point, Chern-Lu formula gives us:

$$\Delta_{\omega'} \log \mathrm{tr}_{\omega'}\omega \geqslant -(n+3)C\,\mathrm{tr}_{\omega'}\omega - \mathrm{tr}_{\omega'} dd^c\psi^+$$

and therefore:

$$\Delta_{\omega'}(\log \mathrm{tr}_{\omega'}\omega + \psi^+) \geqslant -(n+3)C\,\mathrm{tr}_{\omega'}\omega$$

Setting $A = (n+3)C + 1$, we get as usual:

$$\Delta_{\omega'}(\log \mathrm{tr}_{\omega'}\omega + \psi^+ - A\varphi) \geqslant \mathrm{tr}_{\omega'}\omega - nA$$

The end is classic: we choose a point p where $\log \mathrm{tr}_{\omega'}\omega + \psi^+ - A\varphi$ attains its maximum; then we have

$$\begin{aligned} \log \mathrm{tr}_{\omega'}\omega &\leqslant (\log \mathrm{tr}_{\omega'}\omega + \psi^+ - A\varphi)(p) - \psi^+ + A\varphi \\ &\leqslant (\log nA + \sup \psi^+ + 2A \sup |\varphi|) - \psi^+ \end{aligned}$$

which gives the expected result since we have a uniform bound on $||\varphi||_\infty$ as we explained in the beginning of the proof. □

Remark 5.2.2. — Note that it is not clear a priori to deduce the previous estimate using [**Pău08**] by exchanging the role of ω and ω' because we would no longer have control on the bisectional curvature of ω'.

Combining the previous result and Păun's estimate, we obtain the following estimate:

Corollary 5.2.3. — *Let X be a compact Kähler manifold of dimension n, ω, ω' two cohomologous Kähler metrics on X. We assume that $\omega' = \omega + dd^c\varphi$ with $\omega'^n = e^{\psi^+ - \psi^-}\omega^n$ for some smooth functions $\psi^\pm$, and that we have a constant $C > 0$ and some $p > 1$ satisfying:*

(*i*) $\sup_X \psi^+ \leqslant C$ and $||e^{-\psi^-}||_{L^p(\omega^n)} \leqslant C$,

(*ii*) $dd^c\psi^\pm \geqslant -C\omega$.

Then there exists some constant $M > 0$ depending only on n, p, C and ω such that

$$M^{-1}e^{\psi^+}\omega \leqslant \omega' \leqslant Me^{-\psi^-}\omega.$$

5.3. Polystability of the tangent sheaf

Let X be a complex projective variety with log canonical singularities; in particular it is normal, and K_X is $\mathbb{Q}$-Cartier. Its tangent sheaf $\mathscr{T}_X$ is defined as the dual of the sheaf of Kähler differentials; in particular it is reflexive. The main result of this paper is the following:

Theorem 5.3.1. — *Let X be a variety with log canonical singularities such that K_X is ample. Then the sheaf $\mathscr{T}_X$ is polystable with respect to K_X.*

We may deduce the following result, which has already be obtained recently by [**BG13**] and also by [**BHPS12**] using purely algebraic methods:

Corollary 5.3.2. — *Let X be a variety with log canonical singularities such that K_X is ample. Then $H^0(X, \mathscr{T}_X) = 0$.*

Proof. — If we had a non zero global section ξ of $\mathscr{T}_X$, this would induce an injective morphism $\mathscr{O}_X \hookrightarrow \mathscr{T}_X$ given by $f \mapsto f\xi$. By the semi-stability of $\mathscr{T}_X$, this would imply $0 \leqslant -(K_X^n)/n$ which is absurd. □

We will divide the proof of Theorem 5.3.1 into four steps: in the first three, we prove the semi-stability, and in the last one, we refine the result to obtain polystability. As for the first steps, they consists in reducing the statement to some similar one on *smooth* varieties, then dealing with the klt case and finally getting the general case by approximation.

Remark 5.3.3. — Actually the proof of the theorem will yield the semi-stability for lc varieties with *nef and big* canonical bundle. However, unless the augmented base locus $\mathbb{B}_+(K_X)$ has codimension > 1, we will not get the polystability.

5.3.1. Reduction to the log smooth case. — First of all, we reduce the problem on some resolution of X. Indeed, we can choose a resolution $\pi : X' \to X$ which is an isomorphism over X_{reg}, and such that $\pi_* \mathscr{T}_{X'} = \mathscr{T}_X$. To see this, we first observe that for any resolution π leaving the smooth locus untouched, then $\pi_* \mathscr{T}_{X'} \subset \mathscr{T}_X$ by reflexivity of $\mathscr{T}_X$. Then we know from the existence of *canonical* desingularizations (see e.g. [**GKK10**, Cor. 4.7] and the references therein) that we can choose π so that for some exceptional divisor D, $\pi_* \mathscr{T}_{X'}(-\log D) = \mathscr{T}_X$. Combinig these two results, we get the expected identity.

We write $K_{X'} = \pi^* K_X + D$ where D is a snc $\mathbb{Q}$-divisor with coefficients $\geqslant -1$. We set $E = \text{Supp}(D)$, the exceptional set of π. Suppose that we have proved that $\mathscr{T}_{X'}$ is semi-stable with respect to $\pi^* K_X$, and let $\mathscr{F} \subset \mathscr{T}_X$ a coherent subsheaf of $\mathscr{T}_X$ of rank $r > 0$. Over X_{reg}, π is an isomorphism and we can pull back there $\mathscr{F}$ to a coherent sheaf $\mathscr{F}'_0 \subset \mathscr{T}_{|X' \setminus E}$. We extend $\mathscr{F}'$ by saturating it in $\mathscr{T}_{X'}$, ie for any open subset $U \subset X'$, we set $\mathscr{F}'(U) = \mathscr{F}'_0(U \setminus E) \cap \mathscr{T}_{X'}(U)$; this defines a coherent subsheaf of $\mathscr{T}_{X'}$ which has the same rank as $\mathscr{F}$. Now, the $(n-1)$ cycles $\pi_* c_1(\mathscr{F}')$ and $c_1(\mathscr{F})$ agree outside of a set of codimension at least two, so they are equal. By our assumption, $\frac{1}{r}c_1(\mathscr{F}')\cdot(\pi^*K_X)^{n-1} \leqslant \frac{1}{n}c_1(\mathscr{T}_{X'})\cdot(\pi^*K_X)^{n-1}$, so that the projection formula yields

$$\frac{1}{r}c_1(\mathscr{F})\cdot K_X^{n-1} \leqslant \frac{1}{n}c_1(\mathscr{T}_X)\cdot K_X^{n-1}$$

which concludes.

5.3.2. The klt case. — So we change a bit our notations now. We take a log smooth pair (X, D) which is log terminal, we we assume that there exists a birational morphism $\pi : X \to Y$ such that $K_X = \pi^*K_Y + D$, for some $\mathbb{Q}$-Gorenstein normal variety Y with K_Y ample. We also fix an ample divisor A on X, and a Kähler metric $\omega_A \in c_1(A)$. Finally, we write $D = \sum_{i\in I} a_i D_i$, where $a_i > -1$, and we choose defining sections s_i of the smooth hypersurfaces D_i, together with smooth hermitian metrics $|\cdot|_i$ on $\mathcal{O}_X(D_i)$ whose curvature will be denoted by Θ_i. We will follow the strategy of Enoki [**Eno88**] in the canonical case, which consists in working with approximate Kähler-Einstein metrics, and obtaining the slopes inequalities by controlling the error term. In his case, the error term will have the "right" sign, and he does not have to estimate it. This will be our main task. We consider now the following equation, for $\varepsilon, t > 0$:

$$(\omega + t\omega_A + dd^c\varphi)^n = \prod_{i\in I}(|s_i|^2 + \varepsilon^2)^{a_i} e^{\varphi} dV \tag{5.3.1}$$

where $\omega \in c_1(\pi^*K_Y)$ is a semipositive and big form, dV is a volume form whose Ricci curvature is $\mathrm{Ric}\, dV = -\omega - \sum a_i\Theta_i$; we should stress that φ depends of course on ε and t even if we will not make it explicit in the notations. We will set $\omega_\varphi := \omega + t\omega_A + dd^c\varphi$. A classic computation shows that

$$\mathrm{Ric}\,\omega_\varphi = -\omega_\varphi + t\omega_A - \sum_{i\in I} a_i \left(\frac{\varepsilon^2 |D's_i|^2}{(|s_i|^2+\varepsilon^2)^2} + \frac{\varepsilon^2\Theta_i}{|s|^2+\varepsilon^2}\right) \tag{5.3.2}$$

where D' denotes the $(1,0)$-part of the Chern connection induced by $|\cdot|_i$ on $\mathcal{O}_X(D_i)$.

Lemma 5.3.4. — *Let ω_0 be a Kähler metric on X, and ω_φ the unique solution of equation* (5.3.1)*. Then for each index k, the integral*

$$\int_X \frac{\varepsilon^2}{|s_k|^2+\varepsilon^2}\omega_0 \wedge \omega_\varphi^{n-1}$$

converges to 0 *as ε tends to* 0*, $t > 0$ being fixed.*

Proof. — We have $n\omega_0 \wedge \omega_\varphi^{n-1} = \mathrm{tr}_{\omega_\varphi}(\omega_0)\,\omega_\varphi^n$. First note that we may apply Proposition 5.2.1 because we already have a uniform bound on the potential (cf e.g. [**EGZ09**, Theorem 4.1] where the estimate is implicit, or [**Gue12a**, Remark 2.3]). Now, by the definition of ω_φ and Proposition 5.2.1, there exists a constant $C_t > 0$ independent of ε such that:

$$\mathrm{tr}_{\omega_\varphi}(\omega_0)\,\omega_\varphi^n \leqslant C_t\,\frac{\prod_{i\in I}(|s_i|^2+\varepsilon^2)^{a_i}e^{\varphi}dV}{\prod_{a_i>0}(|s_i|^2+\varepsilon^2)^{a_i}}$$

and therefore

$$\frac{\varepsilon^2}{|s_k|^2+\varepsilon^2}\omega_0\wedge\omega_\varphi^{n-1} \leqslant \varepsilon^2\, C'_t\,\frac{\prod_{a_i<0}(|s_i|^2+\varepsilon^2)^{a_i}dV}{|s_k|^2+\varepsilon^2}$$

If the index k is such that $a_k > 0$, then there is almost nothing to check, as $\varepsilon^2(|s_k|^2+\varepsilon^2)^{-1}$ is uniformly integrable in ε. So we may assume that $a_k < 0$ (and of course $a_k > -1$), and that we work on some neighborhood of the divisor. Then, by Fubini theorem, we are boiled to showing that the integral

$$\int_D \frac{\varepsilon^2|dz|^2}{(|z|^2+\varepsilon^2)^{2-\delta}}$$

converges to zero for every $1 > \delta > 0$, where D is the unit disc around 0 in $\mathbb{C}$. Now, performing the change of variable $w = z/\varepsilon$, we get the integral

$$\varepsilon^{2\delta}\int_{|w|^2\leqslant 1/\varepsilon} \frac{|dw|^2}{(1+|w|^2)^{2-\delta}}$$

which goes to zero as $\delta > 0$ and $2-\delta > 1$. □

Recall now the following result, extracted from [**Kob87**] (see also [**Eno88**, Propositions 2.1 & 2.2]), which is the combination of two facts. First, the slope of a subsheaf can be computed (up to increasing it if the sheaf is not reflexive) on the locus where the subsheaf is locally free by integrating there its Chern curvature. And then, the curvature of a subbundle is smaller that the one of the restricted original bundle. More precisely:

Proposition 5.3.5. — *Let $\mathscr{F} \subset \mathcal{O}(E)$ be a coherent subsheaf of positive rank of a hermitian vector bundle (E,h). We let $W(\mathscr{F})$ be the maximal subset of X such that $\mathscr{F}_{X\setminus W(\mathscr{F})}$ defines a holomorphic subbundle $F \subset E_{X\setminus W(\mathscr{F})}$. Then for any Kähler form ω, we have:*

$$n\int_X c_1(\mathscr{F})\wedge\omega^{n-1} \leqslant \int_{X\setminus W(\mathscr{F})} \operatorname{tr}\left(\mathrm{pr}_F\circ\Lambda\Theta_h(E)_{|F}\right)\omega^n$$

We recall that the operator Λ (relatively to ω), initially defined on $(1,1)$ forms α by $\Lambda\alpha := \mathrm{tr}_\omega\alpha$ is naturally extended to bundle-valued $(1,1)$ forms. And what we denote by tr always means the trace of the endomorphism part.

Let us go back to our situation now. We have a coherent subsheaf $\mathscr{F} \subset \mathscr{T}_X$, and we want to compare its slope with the one of $\mathscr{T}_X$ itself. We are going to apply the previous proposition 5.3.5 with ω_φ as reference metric (which will also be used as hermitian metric on T_X). The first step is to understand $\Lambda\Theta_{\omega_\varphi}(T_X)$. We choose geodesic coordinates (z_i) for ω_φ around some point x_0, so that

$$\Theta(T_X)_{x_0} = \sum_{j,k,l,m} R_{j\bar{k}l\bar{m}}dz_j\wedge d\bar{z}_k\otimes\left(\frac{\partial}{\partial z_l}\right)^*\otimes\frac{\partial}{\partial\bar{z}_m}$$

In particular,

$$\Lambda\Theta(T_X)_{x_0} = \sum_{j,l,m} R_{j\bar{j}l\bar{m}}\left(\frac{\partial}{\partial z_l}\right)^*\otimes\frac{\partial}{\partial\bar{z}_m}$$

and using the Kähler symmetry $R_{j\bar{j}l\bar{m}} = R_{l\bar{m}j\bar{j}}$, we find

$$\Lambda\Theta(T_X)_{x_0} = \sum_{j,l,m} R_{l\bar{m}j\bar{j}}\left(\frac{\partial}{\partial z_l}\right)^*\otimes\frac{\partial}{\partial\bar{z}_m}$$

It will be useful to introduce the operator $\sharp$ (relatively to ω) which associates to any $(0,1)$-form α a $(1,0)$ vector $\sharp\alpha$ by

$$\alpha(\bar{u}) = \omega_\varphi(\sharp\alpha, u)$$

for every $u \in T_X$. This operator extends to bundle-valued forms, and one can check easily that if α is a $(1,1)$-form, then $\sharp\alpha$ is an endomorphism of T_X satisfying

$$\mathrm{tr}(\sharp\alpha) = \Lambda\alpha.$$

If we recall that $\mathrm{Ric}\,\omega_\varphi = \sum_{j,l,m} R_{l\bar{m}j\bar{j}} dz_l \wedge d\bar{z}_m$, the computation above can also reformulated as

$$\Lambda\Theta(T_X) = \sharp\mathrm{Ric} \tag{5.3.3}$$

Our metric ω_φ satisfies the equation (5.3.2), ie

$$\mathrm{Ric}\,\omega_\varphi = -\omega_\varphi + t\omega_A - \sum_{i\in I} a_i \left(\frac{\varepsilon^2 |D' s_i|^2}{(|s_i|^2+\varepsilon^2)^2} + \frac{\varepsilon^2 \Theta_i}{|s|^2+\varepsilon^2} \right)$$

It is useful here to separate the "canonical part" from the "log terminal part"; more precisely, we introduce the following notations: $\alpha = \sum_{a_i<0} -a_i \frac{\varepsilon^2|D's_i|^2}{(|s_i|^2+\varepsilon^2)^2}$, $\gamma = \sum_{a_i<0} -a_i \frac{\varepsilon^2\Theta_i}{|s|^2+\varepsilon^2}$, and $\theta = \sum_{i\in I} a_i \left(\frac{\varepsilon^2|D's_i|^2}{(|s_i|^2+\varepsilon^2)^2} + \frac{\varepsilon^2\Theta_i}{|s|^2+\varepsilon^2} \right)$. In particular, we have

$$\mathrm{Ric}\,\omega_\varphi = -\omega_\varphi + t\omega_A - \theta \tag{5.3.4}$$

Moreover, we can choose $C > 0$ big enough so that the rhs of θ and all its summands are dominated by $\chi_\varepsilon\omega_A$, where $\chi_\varepsilon := C\sum_i \varepsilon^2/(|s_i|^2+\varepsilon^2)$.
As α is positive, we have

$$\begin{aligned} \mathrm{tr}\left(\mathrm{pr}_F((\sharp\alpha)_{|F})\right)\omega_\varphi^n &\leqslant \mathrm{tr}(\sharp\alpha)\,\omega_\varphi^n \\ &\leqslant \Lambda\alpha\,\omega_\varphi^n \\ &= n\alpha\wedge\omega_\varphi^{n-1} \end{aligned}$$

and similarly

$$\mathrm{tr}\left(\mathrm{pr}_F((\sharp(\chi_\varepsilon+t)\omega_A)_{|F})\right)\omega_\varphi^n \leqslant n(\chi_\varepsilon+t)\omega_A\wedge\omega_\varphi^{n-1}$$

If we combine the relations (5.3.3) and (5.3.4) with the previous inequalities, we get:

$$\mathrm{tr}\left(\mathrm{pr}_F \circ \Lambda\Theta_{\omega_\varphi}(T_X)_{|F}\right)\,\omega_\varphi^n \leqslant -r\omega_\varphi^n + n(\chi_\varepsilon+t)\omega_A\wedge\omega_\varphi^{n-1} + n\alpha\wedge\omega_\varphi^{n-1}$$

But if follows from equation (5.3.4) that:

$$-\omega_\varphi^n = \mathrm{tr}(\Theta_{\omega_\varphi}(T_X))\wedge\omega_\varphi^{n-1} - t\omega_A\wedge\omega_\varphi^{n-1} + \theta\wedge\omega_\varphi^{n-1}$$

Therefore, Proposition 5.3.5 yields the following inequality:

$$\begin{aligned} \frac{1}{r}\int_X c_1(\mathscr{F})\wedge\omega_\varphi^{n-1} &\leqslant \frac{1}{n}\int_X c_1(T_X)\wedge\omega_\varphi^{n-1} + \left(\frac{1}{r}-\frac{1}{n}\right)\int_X t\omega_A\wedge\omega_\varphi^{n-1} \\ &\quad -\frac{1}{n}\int_X \theta\wedge\omega_\varphi^{n-1} + \frac{1}{r}\int_X \alpha\wedge\omega_\varphi^{n-1} \\ &\quad +\frac{1}{r}\int \chi_\varepsilon\omega_A\wedge\omega_\varphi^{n-1} \end{aligned}$$

Many of these integrals are cohomological, and things get clearer if we write them as intersection numbers. Remembering that $\omega_\varphi \in c_1(\pi^* K_Y + tA)$, $-\theta \in c_1(D)$, we get:

$$\begin{aligned}\mu_{\omega_\varphi}(\mathscr{F}) \quad &\leqslant \quad \mu_{\omega_\varphi}(T_X) + \left(\frac{1}{r} - \frac{1}{n}\right) tA\cdot (\pi^* K_Y + tA)^{n-1} \\ &\quad + \frac{1}{n} D\cdot (\pi^* K_Y + tA)^{n-1} + \frac{1}{r}\int_X \alpha \wedge \omega_\varphi^{n-1} \\ &\quad + \frac{1}{r}\int \chi_\varepsilon \omega_A \wedge \omega_\varphi^{n-1}\end{aligned}$$

where $\mu_{\omega_\varphi}(\mathscr{F})$ (resp. $\mu_{\omega_\varphi}(T_X)$ is the slope of $\mathscr{F}$ (resp. T_X) with respect to ω_φ. In particular, these quantities are independent of ε and go to the respective slopes of $\mathscr{F}$ and T_X relatively to $\pi^* K_Y$ when t converges to 0.

- The quantity $tA\cdot (\pi^* K_Y + tA)^{n-1}$ is independent of ε and converges to 0 when t goes to 0; similarly $D\cdot (\pi^* K_Y + tA)^{n-1}$ is independent of ε and goes to $D\cdot (\pi^* K_Y)^{n-1} = 0$ when t goes to 0.
- As for the term $\int \chi_\varepsilon \omega_A \wedge \omega_\varphi^{n-1}$, we saw in Proposition 5.2.1 that it goes to 0 when $\varepsilon \to 0$, $t > 0$ being fixed.
- Finally, as $\alpha + \gamma \in -c_1(D_-)$ (we set $D_- := \sum_{a_i<0} a_i D_i$), we obtain:

$$\int_X \alpha \wedge \omega_\varphi^{n-1} = -\{D_-\}\cdot (\pi^* K_Y + tA)^{n-1} - \int_X \gamma \wedge \omega_\varphi^{n-1}$$

  but we know that γ is dominated uniformly by $\chi_\varepsilon \omega_A$ so that our integral converges first to $\{D_-\}\cdot (\pi^* K_Y + tA)^{n-1}$ when ε goes to 0, and then to 0 when $t \to 0$.

In conclusion, if we first make ε go to 0, then our inequality becomes purely cohomological, and is continuous with t. And when we evaluate it at $t = 0$, it yields

$$\mu_{\pi^* K_Y}(\mathscr{F}) \leqslant \mu_{\pi^* K_Y}(T_X)$$

which proves the Theorem (in the klt case).

5.3.3. General log canonical case. — Now, (X, D) is log smooth and log canonical. For every $\delta > 0$, $(X, (1-\delta)D)$ is klt, and we have $K_X = \pi^* K_Y + (1-\delta)D + \delta D$. We solve the Monge-Ampère equation (5.3.1) attached to $(X, (1-\delta)D)$, ie:

$$(\omega + t\omega_A + dd^c\varphi)^n = \prod_{i\in I} (|s_i|^2 + \varepsilon^2)^{(1-\delta)a_i} e^\varphi dV \tag{5.3.5}$$

where dV is a volume form whose Ricci curvature is $\operatorname{Ric} dV = -\omega - \sum a_i \Theta_i$. Setting as before $\omega_\varphi := \omega + t\omega_A + dd^c\varphi$ (so ω_φ depends on ε, t and δ), we find:

$$\operatorname{Ric}\omega_\varphi = -\omega_\varphi + t\omega_A - (1-\delta)\sum_{i\in I} a_i \left(\frac{\varepsilon^2 |D' s_i|^2}{(|s_i|^2 + \varepsilon^2)^2} + \frac{\varepsilon^2 \Theta_i}{|s|^2 + \varepsilon^2}\right) - \delta\left(\sum_{i\in I} a_i \Theta_i\right) \tag{5.3.6}$$

We take the same notations as in the previous sections, namely: $\alpha = \sum_{a_i<0} -a_i \frac{\varepsilon^2 |D' s_i|^2}{(|s_i|^2+\varepsilon^2)^2}$, $\theta = \sum_{i\in I} a_i \left(\frac{\varepsilon^2 |D' s_i|^2}{(|s_i|^2+\varepsilon^2)^2} + \frac{\varepsilon^2 \Theta_i}{|s|^2+\varepsilon^2}\right)$, and $\Theta = \sum_{i\in I} a_i \Theta_i$. The preceding equation (5.3.6) becomes:

$$\operatorname{Ric}\omega_\varphi = -\omega_\varphi + t\omega_A - (1-\delta)\theta - \delta\Theta \tag{5.3.7}$$

Choosing a constant C such that $\pm\Theta \leqslant C\omega_A$, we can make the same computations as in the previous paragraph to obtain the inequality

$$\begin{aligned} \frac{1}{r}\int_X c_1(\mathscr{F})\wedge\omega_\varphi^{n-1} \leqslant{} & \frac{1}{n}\int_X c_1(T_X)\wedge\omega_\varphi^{n-1} + \left(\frac{1}{r}-\frac{1}{n}\right)\int_X t\omega_A\wedge\omega_\varphi^{n-1} \\ & -\frac{1-\delta}{n}\int_X \theta\wedge\omega_\varphi^{n-1} + \frac{1-\delta}{r}\int_X \alpha\wedge\omega_\varphi^{n-1} \\ & +\frac{1}{r}\int(\chi_\varepsilon+2C\delta)\omega_A\wedge\omega_\varphi^{n-1} \end{aligned} \tag{5.3.8}$$

If we make first ε and then t go to 0, we find:

$$\mu_{\pi^*K_Y}(\mathscr{F}) \leqslant \mu_{\pi^*K_Y}(T_X) + 2C\delta A\cdot(\pi^*K_Y)^{n-1}$$

and we can conclude by making $\delta\to 0$.

5.3.4. Polystability. — To get polystability, it seems that we really need to know more about the limiting behavior of the approximate Kähler-Einstein metrics ω_φ when ε, t, δ go to zero. Maybe this is why only the semi-stability of $\mathscr{T}_X$ was stated in [**Eno88**] when he treated the case of canonical singularities. But since that time, a lot of work has been done about singular Kähler-Einstein metrics, and we can now use a very recent result obtained by R. Berman and the author [**BG13**] stating that a variety X with log canonical singularities and K_X ample admits a unique Kähler-Einstein metric (of negative curvature) which is smooth on $X_{\rm reg}$. Moreover, this metric is the (weak) limit of its approximations ω_φ constructed above as solution of (5.3.5), and the convergence is smooth on (the compacts of) the regular locus $X_{\rm reg}$.

To be rigorous, the $\mathscr{C}^\infty$ convergence is not showed for the approximations given by (5.3.5) but for some other one leaving the purely log canonical part untouched. But it is not difficult to show that the arguments given in [**BG13**, §4.6-4.7] extend to this different approximation, so we will not prove it here.

Let us now get to the proof of the polystability. The first step, as for semi-stability, is to reduce to the smooth case. So we choose a resolution $\pi: X'\to X$ such that $\pi_*T_{X'} = \mathscr{T}_X$. Suppose now that there exists a proper subsheaf $\mathscr{F}$ of $\mathscr{T}_X$ with the same slope as $\mathscr{T}_X$. Up to taking its reflexive envelop (which does not change the slope), we may assume that $\mathscr{F}$ is already reflexive. As in section 5.3.1, we construct a coherent and *reflexive* subsheaf $\mathscr{F}'\subset T_{X'}$ such that $\pi_*c_1(\mathscr{F}') = c_1(\mathscr{F})$. In particular, we have:

$$\mu_{\pi^*K_X}(\mathscr{F}') = \mu_{\pi^*K_X}(T_{X'}) \tag{5.3.9}$$

Let us introduce some notations to make the picture clearer. We denote by $W' = W(\mathscr{F}')$ the singular set of $\mathscr{F}'$, ie the minimal Zariski closed subset of X' outside which $\mathscr{F}'$ is locally free, given by $\mathcal{O}(F')$ for some vector bundle F' on $X'\setminus W'$. Moreover, we set $E = \mathrm{Supp}(D)$ the exceptional set of the resolution π; this is a purely one codimensional set. Finally, we make the parameters ε, t, δ of the approximation become dependent by choosing $\varepsilon = t = \delta$, and we decide to emphasize this by writing now ω_ε instead of ω_φ. So ω_ε is the approximate Kähler-Einstein metric solution of (5.3.5).

Let us recall now two basic identities (that we gathered to obtain the inequality of Proposition 5.3.5, cf again [**Kob87**] or [**Eno88**, Propositions 2.1 & 2.2]). First, the slope of a reflexive sheaf can be computed (using curvature) on the non-singular:

$$\int_{X'} c_1(\mathscr{F}') \wedge \omega_\varepsilon^{n-1} = \int_{X'\setminus W'} \mathrm{tr}\left(\Theta_{\omega_\varepsilon}(F')\right) \wedge \omega_\varepsilon^{n-1} \tag{5.3.10}$$

Then, if $\beta_\varepsilon \in \mathscr{C}^\infty_{1,0}(X'\setminus W', \mathrm{Hom}(F', F'^{\perp}))$ denotes the second fundamental form of $F' \subset (T_{X'}, \omega_\varepsilon)$, and if β_ε^* is its adjoint (so it is a $(0,1)$ form with values in $\mathrm{Hom}(F'^{\perp}, F')$), we know that $\Theta_{\omega_\varepsilon}(F') = \mathrm{pr}_{F'}(\Theta_{\omega_\varepsilon}(T_X)_{|F'}) + \beta_\varepsilon^* \wedge \beta_\varepsilon$. In particular

$$\begin{aligned} \int_{X'\setminus W'} \mathrm{tr}\left(\Theta_{\omega_\varepsilon}(F')\right) \wedge \omega_\varepsilon^{n-1} &= \frac{1}{n}\int_{X'\setminus W'} \mathrm{tr}\left(\mathrm{pr}_{F'} \circ \Lambda\Theta_{\omega_\varepsilon}(T_X)_{|F'}\right) \omega_\varepsilon^{n} \\ &\quad + \int_{X'\setminus W'} \mathrm{tr}(\beta_\varepsilon^* \wedge \beta_\varepsilon) \wedge \omega_\varepsilon^{n-1} \end{aligned} \tag{5.3.11}$$

Moreover, the $(1,1)$-form $\mathrm{tr}(\beta_\varepsilon^* \wedge \beta_\varepsilon)$ is non-positive, and is identically zero if and only if there is an holomorphic splitting $T_{X'} = F' \oplus F'^{\perp}$ on $X' \setminus W'$.

Using the computations of section 5.3.3, we can deduce from (5.3.11) that:

$$\mu_{\omega_\varepsilon}(\mathscr{F}') \leqslant \mu_{\omega_\varepsilon}(T_{X'}) + \frac{1}{r}\int_{X'\setminus W'} \mathrm{tr}(\beta_\varepsilon^* \wedge \beta_\varepsilon) \wedge \omega_\varepsilon^{n-1} + o(1)$$

Now, as the slopes are continous with ε, and as $\mu_{\pi^*K_X}(\mathscr{F}') = \mu_{\pi^*K_X}(T_{X'})$, we see that

$$0 \leqslant \int_{X'\setminus (W'\cup E)} -\mathrm{tr}(\beta_\varepsilon^* \wedge \beta_\varepsilon) \wedge \omega_\varepsilon^{n-1} \leqslant K(\varepsilon) \tag{5.3.12}$$

for some function $K : \mathbb{R}_+ \to \mathbb{R}_+$ such that $K(\varepsilon)$ goes to 0 when $\varepsilon \to 0$.

Let $\omega_\infty = \lim_{\varepsilon\to 0} \omega_\varepsilon$ be the Kähler-Einstein metric for the pair (X', D), and β_∞ the second fundamental form of $F' \subset T_{X'}$ induced by the Kähler metric ω_∞ on $X' \setminus (W' \cup E)$. By the smooth convergence of ω_ε to ω_∞ on the compacts of $X' \setminus E$, Fatou's lemma applied to (5.3.12) shows that:

$$\int_{X'\setminus (W'\cup E)} \mathrm{tr}(\beta_\infty^* \wedge \beta_\infty) \wedge \omega_\infty^{n-1} = 0$$

so that $\mathrm{tr}(\beta_\infty^* \wedge \beta_\infty)$ and hence β_∞ vanishes on $U' = X' \setminus (W' \cup E)$. In particular, we get an holomorphic splitting $T_{X'} = F' \oplus F'^{\perp}$ on U'.

To be completely rigorous here, we should have kept two parameters, one for ε, and the other one for both t and δ. Indeed, to obtain our inequalities, we first need to make ε go to 0, and then the quantities at stake become cohomological and continuous with t (and δ). However, we can still obtain (5.3.12) in the same way as above but applying Fatou's lemma twice.

We cannot deduce from this much more on the whole X' since the complement of U' has codimension 1, but on X however, things go better. Indeed, on $U := X_{\mathrm{reg}} \setminus W(\mathscr{F})$ (open subset of X whose complement has codimension at least 2), we still have the splitting $\mathscr{T}_X = \mathscr{F} \oplus \mathscr{F}^{\perp}$ with obvious notations (the hermitian structure on $T_{X_{\mathrm{reg}}}$ is given by the

Kähler-Einstein metric which is smooth X_{reg}). So if $j : U \hookrightarrow X$ denotes the open immersion, setting

$$\mathcal{G} := j_* \mathscr{F}^{\perp}$$

defines a reflexive sheaf on X satisfying there

$$\mathscr{T}_X = \mathscr{F} \oplus \mathcal{G}$$

by reflexivity of $\mathscr{T}_X$.

We are almost done. As $\mathscr{T}_X$ is semi-stable and $\mu_{K_X}(\mathscr{F}) = \mu_{K_X}(\mathscr{T}_X)$, we have necessarily $\mu_{K_X}(\mathcal{G}) = \mu_{K_X}(\mathscr{T}_X)$. Then, we run this process with $\mathscr{F}$ chosen to be of smallest rank among the proper non-zero (reflexive) subsheaves of $\mathscr{T}_X$ with same slope. As a consequence, $\mathscr{F}$ is stable, and we can start the process again with $\mathcal{G}$ instead of $\mathscr{T}_X$, which does not affect the previous arguments. This shows the polystability of $\mathscr{T}_X$.

5.4. More stability and generic semipositivity

5.4.1. The case $c_1(K_X) = 0$. — In this section, we prove the following theorem:

Theorem 5.4.1. — *Let X be a compact Kähler space with log canonical singularities. If K_X is numerically trivial, then $\mathscr{T}_X$ is semi-stable with respect to any polarization.*

Proof. — The proof is very similar to the previous one. In order to have unified notations, we will denote by Y our singular initial variety, and we fix some resolution $\pi = X \to Y$. We choose an Kähler class $H \in H^{1,1}(Y, \mathbb{R})$, another Kähler class $A \in H^{1,1}(X, \mathbb{R})$, and we choose a Kähler form $\omega_A \in A$. Similarly, ω will denote a Kähler form on Y belonging to H. As before, we fix parameters $\varepsilon, t, \delta > 0$, but here, we are going to need one more, say σ which is also assumed to be positive. Then, solving the analoguous equation as (5.3.5), we can find a Kähler metric $\omega_\varphi = t(\pi^*\omega + \sigma\omega_A) + dd^c\varphi$ satisfying

$$\operatorname{Ric}\omega_\varphi = -\omega_\varphi + t(\pi^*\omega + \sigma\omega_A) - (1-\delta)\theta - \delta\Theta$$

where $\theta = \sum_{i \in I} a_i \left(\frac{\varepsilon^2 |D' s_i|^2}{(|s_i|^2 + \varepsilon^2)^2} + \frac{\varepsilon^2 \Theta_i}{|s|^2 + \varepsilon^2} \right)$, and $\Theta = \sum_{i \in I} a_i \Theta_i$ with the same notations as in the previous paragraph. The now usual arguments lead to

$$\begin{aligned} \frac{1}{r}\int_X c_1(\mathscr{F}) \wedge \omega_\varphi^{n-1} \leqslant\ & \frac{1}{n}\int_X c_1(T_X) \wedge \omega_\varphi^{n-1} + \left(\frac{1}{r} - \frac{1}{n}\right) \int_X t(\pi^*\omega + \sigma\omega_A) \wedge \omega_\varphi^{n-1} \\ & - \frac{1-\delta}{n} \int_X \theta \wedge \omega_\varphi^{n-1} + \frac{1-\delta}{r} \int_X \alpha \wedge \omega_\varphi^{n-1} \\ & + \frac{1}{r} \int (\chi_\varepsilon + 2C\delta)\omega_A \wedge \omega_\varphi^{n-1} \end{aligned}$$

where as before, $\alpha = \sum_{a_i < 0} -a_i \frac{\varepsilon^2 |D' s_i|^2}{(|s_i|^2 + \varepsilon^2)^2}$. If we make ε go to 0, then we obtain an inequality between intersection numbers, namely

$$\begin{aligned} \frac{t^{n-1}}{r} (c_1(\mathscr{F}) \cdot H_\sigma) \leqslant\ & \frac{t^{n-1}}{n} (c_1(T_X) \cdot H_\sigma) + t^n \left(\frac{1}{r} - \frac{1}{n}\right) (H_\sigma^n) \\ & + (1-\delta) t^{n-1} \left(\frac{1}{n} D \cdot H_\sigma^{n-1} - \frac{1}{r} \{D_-\} \cdot H_\sigma^{n-1} \right) + C\delta t^{n-1} (A \cdot H_\sigma^{n-1}) \end{aligned}$$

where $H_\sigma := \pi^*H + \sigma A$. If we divide by t^{n-1} and let t, δ, σ go to 0, then as D (and D_-) is orthogonal to π^*H, we get

$$\mu_{\pi^*H}(\mathscr{F}) \leqslant \mu_{\pi^*H}(T_X)$$

which concludes.

□

Remark 5.4.2. — We could not get polystability using this process as when the parameters go to 0, the metric ω_φ (or at least its cohomology class) collapses.

5.4.2. Generic semipositivity. — In this last section, we prove the following result:

Theorem 5.4.3. — *Let X be a n-dimensional compact Kähler space with log canonical singularities, and ω a Kähler form. If K_X is nef (resp. $-K_X$ is nef), then Ω^1_X (resp $\mathscr{T}_X$) is generically ω^{n-1}-semipositive.*

Here Ω^1_X denotes the reflexive sheaf of differentials, ie $j_*\Omega^1_{X_{\mathrm{reg}}}$ if $j : X_{\mathrm{reg}} \hookrightarrow X$ is the open immersion, or equivalently Ω^1_X can be defined as the reflexive envelope of the push-forward of the differentials bundle $(\pi_*\Omega_{X'})^{**}$ for any resolution $\pi : X' \to X$. The sheaf $\pi_*\Omega_{X'}$ is already reflexive in the case of log terminal singularities, but it is not the case anymore in the general case of log canonical singularities (cf. [**GKKP11**, Theorem 1.4 & 1.5]).

Recall also that a reflexive sheaf $\mathcal{E}$ on a normal compact Kähler space X endowed with a Kähler form ω of is said to be generically ω-semipositive if for all coherent quotient $\mathscr{F}$ of $\mathcal{E}$, the slope of $\mathscr{F}$ with respect to ω is non-negative, ie $\int_X c_1(\mathscr{F}) \wedge \omega^{n-1} \geqslant 0$. It is well-known that generic semipositivity for all polarizations $(H_1, \ldots, H_{n-1})$ implies the generic nefness.

The case K_X nef of Theorem 5.4.3 is a weak form of Miyaoka semiposivity theorem [**Miy87**], as this celebrated theorem holds for every normal projective variety, any polarization $(H_1, \ldots, H_{n-1})$ and only with the assumption that X is not uniruled, which is automatic if $-K_X$ is nef. However, our result holds in the more general Kähler setting, where it was recently proved in the smooth case by Junyan Cao [**Cao13**]. Let us also mention that it is conjectured [**Pet12**, Conj 1.3] that a projective manifold with nef anticanonical bundle has a generically nef tangent bundle (ie semipositive with respect to every polarization $(H_1, \ldots, H_{n-1})$).

Proof. — Let us begin with the case K_X nef. Here again, as in the previous proofs, we denote by Y the singular original Kähler space, and choose $\pi : X \to Y$ a resolution. We will show that for each coherent subsheaf $\mathscr{F} \subset T_X$, we have $\int_X c_1(\mathscr{F}) \wedge (\pi^*\omega)^{n-1} \leqslant 0$. This will as before show that each coherent subsheaf of $\mathscr{T}_Y$ has non-positive slope with respect to ω, and this will conclude by duality (consider a quotient $\mathcal{G}$ of Ω^1_Y, it induces a subsheaf $\mathcal{G}^* \subset \mathscr{T}_Y$ with non-positive slope, so that $\mathcal{G}$ has a non-negative slope).

We write $K_X = \pi^*K_Y + D$; we know that π^*K_Y is nef, so for every $t > 0$, there exists a smooth form ω_t such that $\omega_t \geqslant -t\omega_A$, where ω_A is a fixed Kähler form on X; we will write $A := \{\omega_A\}$. Let $\omega = \pi^*\omega_Y$ be the pull-back of a Kähler form ω_Y on Y; it is a semipositive big form. Thanks to Yau's theorem, we can find a smooth solution φ of the following Monge-Ampère equation:

$$(\omega + t\omega_A + dd^c\varphi)^n = \prod_{i\in I}(|s_i|^2+\varepsilon^2)^{(1-\delta)a_i}dV_t \tag{5.4.1}$$

where dV_t is a volume form whose Ricci curvature is $\mathrm{Ric}\, dV_t = -\omega_t - \sum a_i\Theta_i$. Setting as before $\omega_\varphi := \omega + t\omega_A + dd^c\varphi$ (so ω_φ depends on ε, t and δ), we find:

$$\mathrm{Ric}\,\omega_\varphi = -\omega_t - (1-\delta)\sum_{i\in I} a_i\left(\frac{\varepsilon^2|D's_i|^2}{(|s_i|^2+\varepsilon^2)^2}+\frac{\varepsilon^2\Theta_i}{|s|^2+\varepsilon^2}\right) - \delta\left(\sum_{i\in I}a_i\Theta_i\right) \tag{5.4.2}$$

We take the same notations as in the previous sections, namely: $\alpha = \sum_{a_i<0} -a_i\frac{\varepsilon^2|D's_i|^2}{(|s_i|^2+\varepsilon^2)^2}$, $\theta = \sum_{i\in I}a_i\left(\frac{\varepsilon^2|D's_i|^2}{(|s_i|^2+\varepsilon^2)^2}+\frac{\varepsilon^2\Theta_i}{|s|^2+\varepsilon^2}\right)$, and $\Theta = \sum_{i\in I}a_i\Theta_i$. The preceding equation (5.4.2) becomes:

$$\mathrm{Ric}\,\omega_\varphi = -\omega_t - (1-\delta)\theta - \delta\Theta \tag{5.4.3}$$

Choosing a constant C such that $\pm\Theta \leqslant C\omega_A$, and remembering that $-\omega_t \leqslant t\omega_A$, we get as before

$$\frac{1}{r}\int_X c_1(\mathscr{F})\wedge\omega_\varphi^{n-1} \leqslant C\left(\int_X \alpha\wedge\omega_\varphi^{n-1} + \int(\chi_\varepsilon+t+\delta)\omega_A\wedge\omega_\varphi^{n-1}\right)$$

If we make first ε and then t go to 0, we find (as $\{\omega_\varphi\} = \{\pi^*\omega_Y\} + tA$ and $\{\pi^*\omega_Y\}$ is orthogonal to any exceptional component) :

$$\mu_{\pi^*\omega_Y}(\mathscr{F}) \leqslant C\delta A\cdot\{\pi^*\omega_Y\}^{n-1}$$

and we can conclude by making $\delta \to 0$.

The case where $-K_X$ is nef is very similar. Again, it is enough to show that every coherent subsheaf $\mathscr{F}\subset\Omega_X^1$ has non-positive slope with respect to any "polarization" ω^{n-1} pulled-back from Y. We solve the same Monge-Ampère equation but now the volume form dV_t satisfies $\mathrm{Ric}\, dV_t = \omega_t - \sum a_i\Theta_i$, where $\omega_t \in c_1(-\pi^*K_Y)$ satisfies $\omega_t \geqslant -t\omega_A$. We get a metric ω_φ satisfying $\mathrm{Ric}\,\omega_\varphi = \omega_t - (1-\delta)\theta - \delta\Theta$. Now, $\Theta(\Omega_X^1) = -\Theta(T_X)^*$, so that $\Lambda\Theta(\Omega_X^1) = -\sharp\mathrm{Ric}$. We can run the same computations as above now, and get the non-positivity of the slope of $\mathscr{F}$. □

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