Positive cones in Kähler geometry

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Contents

1 Cohomology of a compact Kähler manifold

1.1 The de Rham cohomology

Given a differentiable manifold of dimension n and given an integer $0 \leqslant k \leqslant n$, we will denote by $\mathcal{A}^k_M := \Lambda^k T^*_M$ the differentiable vector bundle of smooth k-forms. If the base manifold is fixed, we will sometimes write $\mathcal{A}^k$ instead of $\mathcal{A}^k_M$.

Let us start by recalling the well-known Poincaré lemma on the unit ball in the euclidean space.

Lemma 1.1 (Poincaré lemma)**.** *Let $\alpha \in \mathcal{C}^\infty(B, \mathcal{A}^k)$ be a k-form on the unit ball $B \subset \mathbb{R}^m$ with $k > 0$. If $d\alpha = 0$, then there exists a $(k-1)$-form $\beta \in \mathcal{C}^\infty(B, \mathcal{A}^{k-1})$ such that $\alpha = d\beta$.*

Let now M be a differentiable real manifold of dimension m, let $\mathcal{A}^k_M$ be the $\mathcal{C}^\infty$ vector bundle of differential k forms, and let d be the exterior differential. Closed forms are in general not exact (take $\alpha = d\theta$ on $M = S^1$) and the de Rham cohomology spaces measure this obstruction.

Definition 1.2 (de Rham cohomology)**.** Let M be a differentiable manifold of dimension m. For any integer $0 \leqslant k \leqslant n$, one defines

$$H^k(M, \mathbb{R}) := \{\alpha \in \mathcal{C}^\infty(M, \mathcal{A}^k_M);\ d\alpha = 0\} \Big/ \{d\beta; \beta \in \mathcal{C}^\infty(M, \mathcal{A}^{k-1}_M)\}\ .$$

Remark 1.3 (Comparison with singular cohomology)**.** Consider the following complex

$$\underline{\mathbb{R}}_M \hookrightarrow \mathcal{A}^0_M \xrightarrow{d} \mathcal{A}^1_M \xrightarrow{d} \mathcal{A}^2_M \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^m_M \xrightarrow{d} 0$$

where $\underline{\mathbb{R}}_M$ is the locally constant sheaf on M with values in $\mathbb{R}$, which is naturally a subsheaf of the sheaf of *smooth* 0-forms (i.e. functions). The Poincaré lemma says that this complex is actually a resolution of $\underline{\mathbb{R}}_M$. Since the sheaves $\mathcal{A}^k_M$ are flasque (they admit partitions of unity), the De Rham-Weil isomorphism theorem shows that we have an isomorphism

$$H^k(M, \mathbb{R}) \simeq \check{H}^k(M, \underline{\mathbb{R}}_M)$$

where the RHS is the Čech cohomology in degree k of $\underline{\mathbb{R}}_M$, itself isomorphic to the singular cohomology of M with values in $\mathbb{R}$.

We will later use an explicit construction of that isomorphism in degree 2, as explained below.

Lemma 1.4. *There is an explicit isomorphism*

$$H^2(M, \mathbb{R}) \simeq \check{H}^2(M, \underline{\mathbb{R}}_M) \tag{1.1}$$

Proof. We introduce the subsheaf $\mathcal{Z}^k \subset \mathcal{A}^k$ of smooth, d-closed k-forms on X, so that we have two exact sequences

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathcal{A}^0 \stackrel{d}{\longrightarrow} \mathcal{Z}^1 \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{Z}^1 \longrightarrow \mathcal{A}^1 \stackrel{d}{\longrightarrow} \mathcal{Z}^2 \longrightarrow 0.$$

Since the sheaves $\mathcal{A}^k$ are flasque, they have no cohomology and the following connecting maps are isomorphisms

$$H^1(X, \mathcal{Z}^1) \stackrel{\Phi}{\longrightarrow} \check{H}^2(X, \underline{\mathbb{R}}) \tag{1.2}$$

and

$$H^2(X, \mathbb{R}) = H^0(X, \mathcal{Z}^2)/dH^0(X, \mathcal{A}^1) \stackrel{\Psi}{\longrightarrow} H^1(X, \mathcal{Z}^1) \tag{1.3}$$

In practice, the induced isomorphism $\Phi \circ \Psi : H^2(X, \mathbb{R}) \to \check{H}^2(X, \underline{\mathbb{R}})$ can be described as follows. Let $[\omega] \in H^2(X, \mathbb{R})$, such that $\omega|_{U_\alpha} = dA_\alpha$ for some 1-form A_α and a suitable cover $X = \cup_\alpha U_\alpha$. We get an element $(A_\alpha - A_\beta) \in H^1(X, \mathcal{Z}^1)$, and one writes $A_\alpha - A_\beta = df_{\alpha\beta}$ on $U_{\alpha\beta}$ for some functions $f_{\alpha\beta}$. This defines a 2-cocycle $f_{\alpha\beta\gamma} := f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma}$ with values in $\mathbb{R}$. We thus have $\Phi \circ \Psi([\omega]) = (f_{\alpha\beta\gamma})_{\alpha\beta\gamma}$ where one abusively identified the cocycle with its cohomology class. □

Definition 1.5 (Cup product)**.** The total de Rham cohomology

$$H^\bullet(M, \mathbb{R}) = \bigoplus_{k=0}^{m} H^k(M, \mathbb{R})$$

has a natural ring structure provided by the cup product, which is defined by the wedge product at the level of forms. More precisely, if α, β are two closed forms of degree k and ℓ respectively, then one sets $[\alpha] \cup [\beta] := [\alpha \wedge \beta] \in H^{k+\ell}(M, \mathbb{R})$; which is well-defined thanks to Leibniz formula.

Recall that a connected differentiable manifold M of dimension m is called orientable if there exists a non-vanishing top form $\omega \in \mathcal{C}^\infty(M, \mathcal{A}^m_M)$; that is a trivialization of the line bundle $\mathcal{A}^m_M$. An orientation is a choice of one such form. If M is compact then Stokes formula shows that a trivialization ω is never exact, hence $H^m(M, \mathbb{R}) \neq 0$. More precisely, we have the following duality theorem

Theorem 1.6 (Poincaré duality). *Let M be a connected, orientable differentiable manifold of dimension m and let $0 \leqslant k \leqslant m$ be an integer. Then, the pairing*

$$\begin{aligned} H^k(M,\mathbb{R}) \times H^{m-k}(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ ([\alpha],[\beta]) &\longmapsto \int_M \alpha \wedge \beta \end{aligned}$$

is non-degenerate.

In particular, the integration yields isomorphism $H^m(M,\mathbb{R}) \simeq \mathbb{R}$.

1.2 Basics on currents

Let M be an orientable manifold of dimension m.

Definition 1.7 (Topology on the space $\mathcal{D}_k(M)$). We define $\mathcal{E}_k(M)$ to be the space of smooth k-forms which we equip with the topology induced by the family of seminorms $\|\cdot\|_{\mathcal{C}^r(K)}$ when $K \Subset M$ varies over all compact subsets of M and r varies in $\mathbb{N}$. Of course, one needs to use a family of trivializing charts to define these seminorms, but the induced topology is independent of that choice.

If $K \Subset M$ is a compact subset, one denote by $\mathcal{D}_k(K) \subset \mathcal{E}_k(M)$ the subspace of smooth k-forms with support in K. It is equipped with the induced topology coming from $\mathcal{E}_k(M)$. Finally, the space $\mathcal{D}_k(M) = \cup_{K \Subset M}\mathcal{D}_k(K)$ of smooth k-forms with compact support is equipped with the inductive limit topology.

One can check that a sequence (u_n) in $\mathcal{D}_k(M)$ converges to $u \in \mathcal{D}_k(M)$ if and only if there exists $K \Subset M$ such that for n large enough, u_n is supported on K and given any fixed order $r \in \mathbb{N}$, the coefficients of u_n on any trivializing chart converge in the $\mathcal{C}^r(K)$-topology to those of u. The main point to find a common compact support K for all u_n. There is no harm assuming that $u = 0$, and $k = 0$ for simplicity. If the statement were false, we could find a sequence $x_n \in M$ escaping any compact set such that $u_n(x_n) \neq 0$ (say up to extracting). But then, the set $W := \cap_{n\in\mathbb{N}}\{v \in \mathcal{D}_0(M); |v(x_n)| < |u_n(x_n)|\}$ is a set containing 0 but none of the u_n, and the contradiction comes from the fact that W is open (since $W \cap \mathcal{D}_0(K)$ is a finite intersection of open sets for any given compact set $K \Subset M$).

Definition 1.8 (Currents). A current of degree k is a continuous linear form T on $\mathcal{D}_{m-k}(M)$. We denote by $\mathcal{D}'_k(M)$ the space of currents of degree k on M.

If $T \in \mathcal{D}'_k(M), \alpha \in \mathcal{D}_{m-k}(M)$, we denote by $\langle T, \alpha\rangle \in \mathbb{R}$ the duality pairing. We endow $\mathcal{D}'_k(M)$ with the weak topology, which is the coarsest topology such that for any $\alpha \in \mathcal{D}_{m-k}(M)$, the evaluation map $\ell_\alpha : T \mapsto \langle T, \alpha\rangle$ is continuous. In particular, $T_n \to T$ if and only if for any $\alpha \in \mathcal{D}_{m-k}(M)$, one has $\langle T_n, \alpha\rangle \to \langle T, \alpha\rangle$.

If M is compact and $T \in \mathcal{D}'_m(M) = \mathcal{E}_0(M)^\vee$, we set $\int_M T := \langle T, 1\rangle$.

If $T \in \mathcal{D}'_k(M)$, we define $dT \in \mathcal{D}'_{k+1}(M)$ by setting $\langle dT, \alpha\rangle = (-1)^k\langle T, d\alpha\rangle$ for any $\alpha \in \mathcal{D}_{m-k-1}(M)$.

We have a natural injection

$$j : \mathcal{E}_k(M) \hookrightarrow \mathcal{D}'_k(M)$$

defined by $\langle j(\alpha), \beta\rangle := \int_M \alpha \wedge \beta$, for any $\beta \in \mathcal{D}_{m-k}(M)$. If M is compact and $\alpha \in \mathcal{E}_m(M)$, then $\int_M \alpha = \langle j(\alpha), 1\rangle = \int_M j(\alpha)$.

If $T \in \mathcal{D}'_k(M), \beta \in \mathcal{E}_\ell(M)$, we define $T \wedge \beta \in \mathcal{D}'_{k+\ell}(M)$ by $\langle T \wedge \beta, \alpha\rangle := \langle T, \beta \wedge \alpha\rangle$ for any $\alpha \in \mathcal{D}_{m-k-\ell}(M)$. It $T = j(\alpha)$ for $\alpha \in \mathcal{E}_k(M)$, then $j(\alpha) \wedge \beta = j(\alpha \wedge \beta)$. Moreover, if M is compact and $\beta \in \mathcal{E}_{m-k}(M)$, then $T \wedge \beta \in \mathcal{E}_0(M)^\vee$ and we have $\int_M T \wedge \beta = \langle T \wedge \beta, 1\rangle = \langle T, \beta\rangle$ essentially by definition of these quantities.

Similarly to de Rham cohomology for forms, we have a de Rham cohomology for currents, say $H^k_{cur}(M, \mathbb{R})$ and a natural linear morphism

$$J : H^k(M, \mathbb{R}) \longrightarrow H^k_{cur}(M, \mathbb{R}).$$

One can prove a Poincaré lemma for currents, as well as a regularization result for closed current T which imply that the above map is an isomorphism.

Lemma 1.9. *Assume that M is compact, and let α (resp. β) be a smooth closed k-form (resp. closed $(m-k)$-form). Let $T \in \mathcal{D}'_k(M)$ be a closed k-current such that $T \in [\alpha]$ under the isomorphism J. Then we have*

$$\int_M T \wedge \beta = \langle T, \beta\rangle = [\alpha] \cup [\beta]$$

Proof. By assumption, $T = j(\alpha) + dS$ for some $(k-1)$-current S. Since β is closed, we have $\langle T, \beta\rangle = \langle j(\alpha), \beta\rangle = \int_M \alpha \wedge \beta$. □

Proposition 1.10 (Continuity of cohomology map)**.** *Let M be a compact orientable manifold. Then, then map*

$$\mathcal{D}'_k(M) \longrightarrow H^k_{cur}(M, \mathbb{R})$$
$$T \longmapsto [T]$$

is continuous for the weak topology.

Proof. By Poincaré duality and Lemma 1.9, one can find a basis $[T_i]$ (resp. $[u_i]$) of $H^k_{cur}(M, \mathbb{R})$ (resp. $H^{n-k}(M, \mathbb{R})$) such that $\int_M T_i \wedge u_j = \delta_{ij}$. Now, if T is a closed k-current, one can uniquely decompose $[T] = \sum a_i[T_i]$ and one has $a_i = [T] \cup [u_i] = \int_M T \wedge u_i$. In particular, the coefficients a_i depend continuously on T. □

Definition 1.11 (Current of integration)**.** Let M be an orientable manifold of dimension m and let $Z \subset M$ be a closed submanifold of dimension k . We define the current of integration $[Z]$ on M to be the current of degree $(m-k)$ such that if $\alpha \in \mathcal{C}^\infty_c(M, \mathcal{A}^k_M)$, then

$$\langle [Z], \alpha\rangle := \int_Z \alpha$$

where the last integral is defined to be $\int_Z j^*\alpha$ if $j : Z \hookrightarrow M$ is the natural closed immersion. If M is compact, then $[Z]$ is closed by Stokes formula.

1.3 Complex-valued differential forms

Let X be a complex manifold of dimension n. We denote by T_X (resp. Ω_X) the holomorphic tangent bundle (resp. holomorphic cotangent bundle) of X.

Definition 1.12 (The complex vector bundles $T_X^{1,0}$ and $T_X^{0,1}$)**.** The real tangent bundle $T_{X,\mathbb{R}}$, i.e. the tangent bundle of X seen as a real differentiable manifold, is a $\mathcal{C}^\infty$ real vector bundle with rank $2n$ endowed with an action $J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ induced by the multiplication by i on T_X; it satisfies $J^2 = -\mathrm{Id}_{T_{X,\mathbb{R}}}$. We decompose the complexified tangent bundle $T_{X,\mathbb{C}} := T_{X,\mathbb{R}} \otimes \mathbb{C}$

$$T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1} \tag{1.4}$$

according to its eigenspaces, where $T_{X,x}^{1,0} := \{u \in (T_{X,\mathbb{C}})_x; Ju = iu\}$ and $T_{X,x}^{0,1} := \{u \in (T_{X,\mathbb{C}})_x; Ju = -iu\}$ is the complex conjugate of $T_{X,x}^{1,0}$.

The canonical realization of the *holomorphic* tangent bundle T_X inside $T_{X,\mathbb{R}} \otimes \mathbb{C}$ is isomorphic to $T_X^{1,0}$ (as $\mathcal{C}^\infty$ complex vector bundles).

Local picture. Under a local trivialization of $X \supset U \simeq \mathbb{C}^n$, we have complex coordinates $z_1, \ldots, z_n$ as well as real coordinates $x_k = \mathrm{Re}(z_k), y_k = \mathrm{Im}(z_k)$ which induce vector fields $\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k} \in T_{X,\mathbb{R}}$ and $\frac{\partial}{\partial z_k} \in T_X$. Then $T_{X,\mathbb{R}} := \oplus_{k=1}^n (\mathbb{R}\frac{\partial}{\partial x_k} \oplus \mathbb{R}\frac{\partial}{\partial y_k})$ and the operator J satisfies $J\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}$ and $J\frac{\partial}{\partial y_k} = -\frac{\partial}{\partial x_k}$. Since $\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)$, we find $J\frac{\partial}{\partial z_k} = i\frac{\partial}{\partial z_k}$. The conjugation operator $T_{X,\mathbb{R}} \otimes \mathbb{C} \ni v \otimes \lambda \mapsto v \otimes \bar{\lambda}$ is a $\mathbb{C}$-antilinear automorphism such that $\overline{\frac{\partial}{\partial z_k}} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) =: \frac{\partial}{\partial \bar{z}_k}$ satisfies $J\frac{\partial}{\partial \bar{z}_k} = -i\frac{\partial}{\partial \bar{z}_k}$. In particular, we find that over U, $T_X^{1,0} = \oplus_{k=1}^n \mathbb{C}\frac{\partial}{\partial z_k}$ and $T_X^{0,1} = \oplus_{k=1}^n \mathbb{C}\frac{\partial}{\partial \bar{z}_k}$, and finally, $\overline{T_X^{1,0}} = T_X^{0,1}$.

We define the complex vector bundles $\mathcal{A}_{X,\mathbb{C}} := T_{X,\mathbb{C}}^*, \mathcal{A}_X^{1,0} := (T_X^{1,0})^*$, and similarly $\mathcal{A}_X^{0,1} := (T_X^{0,1})^*$. Dualizing (1.4), one gets

$$\mathcal{A}_{X,\mathbb{C}} = \mathcal{A}_X^{1,0} \oplus \mathcal{A}_X^{0,1} \tag{1.5}$$

and similarly as for the tangent, we see that $\mathcal{A}_X^{1,0}$ is spanned by the 1-forms $dz_k := \frac{1}{2}(dx_k + idy_k)$ and $\Omega_X^{0,1}$ is spanned by the 1-forms $d\bar{z}_k := \frac{1}{2}(dx_k - idy_k)$.

Finally, one defines for any integer $1 \leqslant k \leqslant n$ the following complex vector bundle $\mathcal{A}_{X,\mathbb{C}}^k := \Lambda^k \mathcal{A}_{X,\mathbb{C}}$ as well as for any $1 \leqslant p, q \leqslant n$, $\mathcal{A}_X^{p,q} := \Lambda^p \mathcal{A}_X^{1,0} \otimes \Lambda^q \mathcal{A}_X^{0,1}$. One gets a decomposition

$$\mathcal{A}_{X,\mathbb{C}}^k = \bigoplus_{p+q=k} \mathcal{A}_X^{p,q}. \tag{1.6}$$

Locally, a smooth (p,q)-form α can be uniquely represented as $\alpha = \sum_{I,J} f_{IJ} dz_I \wedge d\bar{z}_J$ where the sum runs over all ordered subsets $I, J \subset \{1, \ldots, n\}$ with p (resp. q) elements, and f_{IJ} are smooth, complex-valued functions. Moreover, if $I = \{i_1, \ldots, i_p\}$, one writes $dz_I := dz_{i_1} \wedge \ldots \wedge dz_{i_p}$ and similarly for $d\bar{z}_J$.

Definition 1.13 (Real forms)**.** A form $\alpha \in \mathcal{C}^\infty(X, \mathcal{A}_X^k)$ is real if $\alpha = \bar{\alpha}$. If, moreover, $\alpha \in \mathcal{C}^\infty(X, \mathcal{A}_X^{p,q})$, then $p = q$ unless $\alpha = 0$. In local coordinates, a (p, p)-form $\alpha = \sum_{I,J} f_{IJ} dz_I \wedge d\bar{z}_J$ is real if and only if $f_{IJ} = -\overline{f_{JI}}$.

Definition 1.14 (Kähler metrics, Kähler manifolds)**.** Let X be a complex manifold. A hermitian metric on X is a hermitian positive definite form of class $\mathcal{C}^\infty$ on T_X. In local coordinates, $h = \sum_{k,\ell} h_{k\ell} dz_k \otimes d\bar{z}_\ell$. One associate to h the fundamental form $\omega := -\mathrm{Im}(h)$. In local coordinates, $\omega = \frac{i}{2} \sum_{k\ell} h_{k\ell} dz_k \wedge d\bar{z}_\ell$. It is real, of type $(1, 1)$. On says that ω is Kähler if $d\omega = 0$. Finally, X is a Kähler manifold if it admits a Kähler metric.

1.4 The Dolbeault cohomology

The exterior differential $d : \mathcal{A}_{X,\mathbb{R}}^k \to \mathcal{A}_{X,\mathbb{R}}^{k+1}$ extends by $\mathbb{C}$-linearity to $d : \mathcal{A}_{X,\mathbb{C}}^k \to \mathcal{A}_{X,\mathbb{C}}^{k+1}$. Moreover, if α is a smooth (p, q)-form, it is clear from Leibniz rule that one can uniquely decompose $d\alpha = (d\alpha)_{p+1,q} + (d\alpha)_{p,q+1}$ according to its type. We define $\partial\alpha$ (resp. $\bar{\partial}\alpha$) to be the $(p + 1, q)$-component (resp. $(p, q + 1)$) of $d\alpha$. In local coordinates, we get the following expression for ∂ and $\bar{\partial}$:

$$\partial\alpha = \sum_{I,J} \sum_{k=1}^{n} \frac{\partial f_{IJ}}{\partial z_k} dz_k \wedge dz_I \wedge d\bar{z}_J$$

$$\bar{\partial}\alpha = \sum_{I,J} \sum_{k=1}^{n} \frac{\partial f_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J$$

In particular, a $(p, 0)$-form $\alpha \in \mathcal{C}^\infty(X, \mathcal{A}_{X,\mathbb{C}}^{p,0})$ is holomorphic if and only if $\bar{\partial}\alpha = 0$.

Clearly, we have $d = \partial + \bar{\partial}$. Since $d^2 = 0$, we infer

$$\partial^2 = \bar{\partial}^2 = 0 \quad \text{and} \quad \partial\bar{\partial} = -\bar{\partial}\partial.$$

In particular, a $\bar{\partial}$-exact form is $\bar{\partial}$-closed (and the same holds with ∂). While the converse is certainly not true (take $\alpha = \frac{d\bar{z}}{\bar{z}}$ on $\mathbb{C}^*$), it is true *locally*. More precisely, we have the complex analogue of Poincaré lemma, cf [Voi02, Proposition 2.31].

Lemma 1.15 (Dolbeault-Grothendieck lemma)**.** *Let α be a (p, q)-form on the unit polydisk $D \subset \mathbb{C}^n$ with $q > 0$. If $\bar{\partial}\alpha = 0$, then there exists a $(p, q - 1)$-form β on D such that $\alpha = \bar{\partial}\beta$.*

Definition 1.16 (Dolbeault cohomology)**.** Let X be a complex manifold of dimension n. For any indices $0 \leqslant p, q \leqslant n$, one defines

$$H^{p,q}(X) := \{\alpha \in \mathcal{C}^\infty(X, \mathcal{A}_X^{p,q});\ \bar{\partial}\alpha = 0\} \Big/ \{\bar{\partial}\beta; \beta \in \mathcal{C}^\infty(X, \mathcal{A}_X^{p,q-1})\}\ .$$

At this point, the $\mathbb{C}$-vector spaces $H^{p,q}(X)$ could be infinite dimensional. For instance, $H^{0,0}(X) = \mathcal{O}_X(X)$ is the space of holomorphic functions on X.

Remark 1.17 (Comparison with coherent cohomology)**.** Consider the following complex

$$\Omega_X^p \hookrightarrow \mathcal{A}_X^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}_X^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}_X^{p,2} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}_X^{p,n} \xrightarrow{\bar{\partial}} 0$$

where Ω_X^p is the sheaf of *holomorphic* p-forms, which is naturally a subsheaf of the sheaf of *smooth* $(p,0)$-forms. The Dolbeault-Grothendieck lemma says that this complex is actually a resolution of Ω_X^p. Since the sheaves $\mathcal{A}_X^{p,q}$ are flasque (they admit partitions of unity), the De Rham-Weil isomorphism theorem shows that we have an isomorphism

$$H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$$

where the RHS is the coherent cohomology in degree q of Ω_X^p (or, equivalenty, the Čech cohomology of that sheaf).

1.5 The Hodge decomposition theorem

When (X,g) is a *compact* Riemannian manifold, then Hodge theory shows that the de Rham cohomology

$$H^k(X,\mathbb{R}) := \{\alpha \in \mathcal{C}^\infty(X, \mathcal{A}_X^k);\ d\alpha = 0\} \Big/ \{d\beta; \beta \in \mathcal{C}^\infty(X, \mathcal{A}_X^k)\}$$

is isomorphic to the space $\mathcal{H}_g^k(X)$ of harmonic k-forms (i.e. k-forms α such that $\Delta_d \alpha = 0$, where $\Delta_d = dd^* + d^*d$ is the Hodge laplacian associated to g). In other words, each de Rham cohomology class contains a unique harmonic representative. A first consequence is that $H^k(X)$ is finite dimensional. As another application, one can use this correspondence to prove the Poincaré duality, cf [Voi02, Theorem 5.30]

Let us now assume that X admits a complex structure J such that $J \in O(g)$ and $\omega := g(\cdot, J\cdot)$ is closed; hence ω is a Kähler form. We can consider several laplacians, among which $\Delta_d = dd^* + d^*d$, $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ and $\Delta_\partial = \partial\partial^* + \partial^*\partial$. The last two laplacians have the advantage of preserving the type, i.e. if $\alpha \in \mathcal{C}^\infty(X, \mathcal{A}_X^{p,q})$, then $\Delta_{\bar\partial}\alpha \in \mathcal{C}^\infty(X, \mathcal{A}_X^{p,q})$. In particular, usual Hodge theory shows that we have an isomorphism between $H^{p,q}(X)$ and the space $\mathcal{H}_{\bar\partial}^{p,q}(X)$ of $\Delta_{\bar\partial}$-harmonic (p,q)-forms.

One of the main results in Hodge theory states that when ω is Kähler, then we have the following identity between the real and complex laplacian $\frac{1}{2}\Delta_d = \Delta_{\bar\partial} = \Delta_\partial$. In particular, we have $\overline{\Delta_{\bar\partial}} = \Delta_\partial = \Delta_{\bar\partial}$ hence $H_{\bar\partial}^{p,q}(X) \simeq \overline{H_{\bar\partial}^{q,p}(X)}$. Moreover if α is a complex-valued Δ_d-harmonic k-form and $\alpha = \sum_{p+q=k} \alpha_{p,q}$ is its type decomposition, then $\alpha_{p,q}$ is $\Delta_{\bar\partial}$-harmonic as well. This yields the following foundational result

Theorem 1.18 (Hodge decomposition theorem)**.** *Let X be a compact Kähler manifold.There are isomorphisms*

$$H^k(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X)$$

$$H^{q,p}(X) \simeq \overline{H^{q,p}(X)}.$$

One can even refine the statement making the isomorphisms canonical (i.e. independent of the choice of a Kähler metric).

One should insist on the fact that the decomposition above in cohomology does not arise directly from the decomposition $\mathcal{A}_{X,\mathbb{C}}^k = \oplus_{p+q=}\mathcal{A}_X^{p,q}$. For instance, consider $\alpha =$

$\bar{z}_2 dz_1 + z_1 d\bar{z}_2$. Then α is closed ($d\alpha = 0$) but its components are not $\bar{\partial}$-closed ($\bar{\partial}(\bar{z}_2 dz_1) = -dz_1 \wedge d\bar{z}_2 \neq 0$).

For $p = q$, $H^{p,p}(X) \subset H^2(X, \mathbb{C})$ is a complex vector space which is stable under conjugation (recall that $H^2(X,\mathbb{C}) = H^2(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ by the universal coefficient theorem). In particular, there is a real vector space $H^{p,p}(X,\mathbb{R}) \subset H^{p,p}(X)$ such that $H^{p,p}(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. One can identify $H^{p,p}(X,\mathbb{R})$ with $H^{p,p}(X) \cap H^2(X,\mathbb{R})$, i.e. it consists of (p,p)-classes that can be represented by real, $\bar{\partial}$-closed (p,p)-forms.

2 Line bundles

2.1 First Chern class of a line bundle

Let X be a complex manifold of dimension n.

Definition 2.1 (Line bundle)**.** A line bundle L on X consists of a complex manifold L endowed with a projection map $p : L \to X$ such that X admits an open covering $X = \cup U_\alpha$ and isomorphisms

$$\begin{array}{ccc} L|_{p^{-1}(U_\alpha)} & \xrightarrow[\simeq]{\tau_\alpha} & U_\alpha \times \mathbb{C} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & U_\alpha & \end{array}$$

such that on the overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$, the isomorphism $\tau_\alpha \circ \tau_\beta^{-1} : U_{\alpha\beta} \times \mathbb{C} \to U_{\alpha\beta} \times \mathbb{C}$ is given by $(x, v) \mapsto (x, g_{\alpha\beta}(x)v)$ for some $g_{\alpha\beta} \in \mathcal{O}_X(U_{\alpha\beta})^*$.

If $x \in X$, we denote by $L_x := p^{-1}(x)$ the fiber of p at x; which is non-canonically isomorphic to $\mathbb{C}$. Up to refining the cover, one can assume that the sets U_α are isomorphic to a polydisk in $\mathbb{C}^n$, and that the double overlaps $U_{\alpha\beta}$ are simply connected.

Definition 2.2 (Sections)**.** If $U \subset X$, a smooth (resp. holomorphic) section of L over U is a smooth (resp. holomorphic) map $s : U \to L$ such that $p(s(x)) = x$ for all $x \in U$. That is, $s(x) \in L_x$ for all $x \in U$. We write $s \in \mathcal{C}^\infty(U, L)$ (resp. $s \in H^0(U, L)$).

If s is a section of L over X, then $\tau_\alpha(s(x)) = (x, \sigma_\alpha(x))$ for some *function* $\sigma_\alpha : U_\alpha \to \mathbb{C}$. On the overlap $U_{\alpha\beta}$, one has $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$. Conversely, the data of functions $\sigma_\alpha : U_\alpha \to \mathbb{C}$ satisfying the relation $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$ on the overlaps induces a unique section s of L corresponding to the σ_α under the trivialization.

Definition 2.3 (Meromorphic sections)**.** A meromorphic section of a line bundle L on X consists of the data of meromorphic functions $\sigma_\alpha \in \mathcal{M}_X(U_\alpha)$ such that $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$ on every overlap $U_{\alpha\beta}$.

Let us add a few of remarks:

• Global holomorphic sections $s \in H^0(X, L)$ may not exist, although $\mathcal{C}^\infty(X, L)$ is an infinite dimensional vector space, since once can construct many smooth sections using partitions of unity or even just cut-off functions.

• Locally, the map τ_α induces a non-vanishing, holomorphic section $x \mapsto e_\alpha(x) := \tau_\alpha^{-1}(x, 1)$ called local trivialization of L on U_α.

• If s, s' are two meromorphic sections of L, then the quotient $\frac{s}{s'}$ defines a meromorphic *function* on X (provided $s' \not\equiv 0$).

Definition 2.4 (The Picard group of X)**.** Two line bundles L, L' on X are isomorphic if there exists a biholomorphic map $f : L \to L'$ commuting with the projections to X and such that for any $x \in X$, the restriction $f|_{L_x} : L_x \to L'_x$ is a linear isomorphism. We denote by $\mathrm{Pic}(X)$ the space of isomorphism classes of line bundles on X. Tensor product of line bundles induces an abelian group structure on $\mathrm{Pic}(X)$.

2.1.1 The cocycle point of view

The data of $(U_{\alpha\beta}, g_{\alpha\beta})$ characterizes L up to isomorphism of line bundles. Moreover, one has the following relation on $U_{\alpha\beta\gamma}$: $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$. In particular, L defines a unique cocycle in $H^1(X, \mathcal{O}_X^*)$ and the map that sends L to that cocycle induces a group isomorphism

$$\mathrm{Pic}(X) \xrightarrow{\simeq} H^1(X, \mathcal{O}_X^*).$$

The exponential exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_X \xrightarrow{2\pi i\cdot} \mathcal{O}_X \xrightarrow{\exp(\cdot)} \mathcal{O}_X^* \longrightarrow 0$$

induces a long exact sequence in cohomology and, in particular, a map

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{\psi} \check{H}^2(X, \underline{\mathbb{Z}}_X)$$

that can be described as follows. If $(U_{\alpha\beta}, g_{\alpha\beta})_{\alpha\beta}$ is a cocycle with values in $\mathcal{O}_X^*$, and since $U_{\alpha\beta}$ is simply connected, there exists $f_{\alpha\beta} \in \mathcal{O}_X(U_{\alpha\beta})$ satisfying

$$e^{2\pi i f_{\alpha\beta}} = g_{\alpha\beta}. \tag{2.7}$$

On the triple overlap $U_{\alpha\beta\gamma}$, we have $c_{\alpha\beta\gamma} := f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma} \in \underline{\mathbb{Z}}_X(U_{\alpha\beta\gamma})$, and $(U_{\alpha\beta\gamma}, c_{\alpha\beta\gamma})_{\alpha\beta\gamma}$ defines an element in $H^2(X, \underline{\mathbb{Z}}_X)$. Finally, the inclusion map $\underline{\mathbb{Z}}_X \hookrightarrow \underline{\mathbb{R}}_X$ induces a map $j : \check{H}^2(X, \underline{\mathbb{Z}}_X) \to \check{H}^2(X, \underline{\mathbb{R}}_X)$. One defines the first Chern class of a line bundle L as the image of the composite map $c_1 := j \circ \psi$

$$\mathrm{Pic}(X) \xrightarrow{\psi} \check{H}^2(X, \underline{\mathbb{Z}}) \xrightarrow{j} \check{H}^2(X, \underline{\mathbb{R}}), \qquad c_1 = j\circ\psi \tag{2.8}$$

where one has used the identification $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*)$.

2.1.2 The metric point of view

Definition 2.5 (Hermitian metric)**.** Let $L \to X$ be a line bundle on a complex manifold X. A hermitian metric h on L is a collection of hermitian metrics $(h_x)_{x\in X}$ on L_x that vary smoothly with x.

Concretely, h can be viewed in the trivializations $L|_{p^{-1}(U_\alpha)} \simeq U_\alpha \times \mathbb{C}$ as a map $(x, \lambda) \mapsto |\lambda|^2 e^{-\phi_\alpha}$ where $\phi_\alpha : U_\alpha \to \mathbb{R}_{>0}$ is a smooth function. The choice to write a positive function as an exponential is of course arbitrary, but we will see later that it is particularly convenient. Alternatively, one can use the trivialization $e_\alpha \in H^0(U_\alpha, L)$ to describe h in the chart U_α by setting $|e_\alpha|_h^2 := e^{-\phi_\alpha}$. The functions $\phi_\alpha \in \mathscr{C}^\infty(U_\alpha, \mathbb{R})$ are called local weights of h; of course they depend on the choice of local trivializations of L.

The functions ϕ_α do not match on the overlap $U_{\alpha\beta}$ in general, but since $e_\beta = g_{\alpha\beta} e_\alpha$, we find the relation $\phi_\alpha - \phi_\beta = \log |g_{\alpha\beta}|^2$. Since $g_{\alpha\beta}$ is holomorphic and non-vanishing on $U_{\alpha\beta}$, we have $\partial\bar\partial\phi_\alpha = \partial\bar\partial\phi_\beta$ on $U_{\alpha\beta}$.

Conversely, if L is given, one can construct a hermitian metric h on L by setting $|e_\alpha|_h^2 := \sum_\gamma \chi_\gamma |g_{\gamma\alpha}|^2$ where χ_γ is a partition of unity subordinate to $(U_\gamma)_\gamma$. In other words, one set

$$\phi_\alpha := -\log \sum_\gamma \chi_\gamma |g_{\gamma\alpha}|^2. \tag{2.9}$$

Given the relation $e_\beta = g_{\alpha\beta} e_\alpha$, h is well-defined if and only if we have $|e_\beta|_h^2 = |g_{\alpha\beta}|^2 |e_\alpha|_h^2$ on the overlap. This is in turn equivalent to

$$\phi_\alpha - \phi_\beta = \log |g_{\alpha\beta}|^2 \tag{2.10}$$

which follows immediately from the cocyle relation $g_{\gamma\alpha} g_{\alpha\beta} = g_{\gamma\beta}$.

Definition 2.6 (Chern curvature form)**.** Let h be a hermitian metric on a line bundle $L \to X$. The Chern curvature form $\Theta_h(L)$ is the real, closed $(1,1)$-form defined on U_α by the formula $\Theta_h(L) := \frac{i}{2\pi}\partial\bar\partial\phi_\alpha$. It only depends on h and not on the trivializations of L.

Since $\Theta_h(L)$ is locally exact, it is closed. In particular, it defines an element $[\Theta_h(L)] \in H^2(X, \mathbb{R})$. Moreover, if h, h' are two hermitian metrics on a line bundle $L \to X$, then it follows directly from the definition that there exists a smooth *function* f on X such that $h' = e^f h$. In particular, one has $\Theta_h(L) = \Theta_{h'}(L) + \frac{i}{2\pi}\partial\bar\partial f$, hence $[\Theta_h(L)] = [\Theta_{h'}(L)] \in H^{1,1}(X)$ and we have a well-defined cohomology class $c_1(L) \in H^2(X, \mathbb{R})$.

Lemma 2.7. *The "metric" and "cocycle" definitions of $c_1(L)$ coincide under the identification* (1.1) *between $H^2(X, \mathbb{R})$ and $\check{H}^2(X, \underline{\mathbb{R}})$.*

Proof. Let $c_1^m(L) \in H^2(X, \mathbb{R})$ (resp. $c_1^c(L) \in \check{H}^2(X, \underline{\mathbb{R}})$) be the "metric" (resp. "cocycle") first Chern class of L. We borrow the notation Ψ and Φ from the proof of the isomorphism (1.1).

Recall that $c_1^m(L)$ can be represented by the Chern curvature $\Theta_h(L)$ of any hermitian metric h, e.g. the one defined by (2.9). In that case, we have on U_α the identity $\Theta_h(L) = \frac{i}{2\pi} d\bar\partial\phi_\alpha$, so that $\Psi(c_1^m(L))$ is represented by the cocyle

$$(-\frac{i}{2\pi}\partial(\phi_\alpha - \phi_\beta))_{\alpha\beta} = (-\frac{i}{2\pi} g_{\alpha\beta}^{-1} d g_{\alpha\beta})_{\alpha\beta} \in H^1(X, \mathcal{Z}^1)$$

thanks to (2.10). Now remember from (2.7) that $-\frac{i}{2\pi} g_{\alpha\beta}^{-1} d g_{\alpha\beta} = d f_{\alpha\beta}$ and it follows from the definition of $c_1^c(L)$ that $\Phi \circ \Psi(c_1^m(L)) = c_1^c(L)$. □

Definition 2.8 (Singular hermitian metric)**.** Let $L \to X$ be a line bundle on a complex manifold X. A singular hermitian metric h on L is defined as $h = e^{-f} h_0$ where $f \in L^1_{\mathrm{loc}}(X)$ and h_0 is a smooth hermitian metric on L.

The curvature of h is defined by $\Theta_h(L) = \Theta_{h_0}(L) + \frac{i}{2\pi}\partial\bar{\partial} f$; it is a closed $(1,1)$ current whose cohomology class is $c_1(L)$.

Lemma 2.9. *Let $L \to X$ be a line bundle on a compact Kähler manifold X and let $\alpha \in c_1(L)$ be a closed $(1,1)$-form (resp. a closed $(1,1)$-current). Then, there exists an hermitian metric (resp. singular hermitian metric) h on L such that $\Theta_h(L) = \alpha$.*

Proof. Let h_0 be some background hermitian metric on L. The cohomology class of the closed $(1,1)$-form (or current) $\alpha - \Theta_{h_0}(L)$ is zero, hence by the $\partial\bar{\partial}$-lemma (cf Lemma 3.15 below), there exists a function $f \in \mathcal{C}^\infty(X)$ (resp. $f \in L^1(X)$) such that $\alpha - \Theta_{h_0}(L) = \frac{i}{2\pi}\partial\bar{\partial} f$. Set $h := e^{-f} h_0$; it satisfies the requirements. □

2.1.3 Lefschetz theorem on $(1,1)$-classes

We have seen in the previous section (cf Lemma 2.7 and (2.8)) that if $L \to X$ is a complex manifold, then $c_1(L) \in H^2(X, \mathbb{R})$ lies in the image of $H^2(X, \underline{\mathbb{Z}}) \to H^2(X, \underline{\mathbb{R}})$. Moreover, it is clear from the metric definition of $c_1(L)$ that it can be represented by a (d-closed) $(1,1)$-form. Actually, the converse holds.

More precisely, we say that a class $\alpha \in H^2(X, \mathbb{R})$ is an *integral class* if under the isomorphism (1.1) between $H^2(X, \mathbb{R})$ and $H^2(X, \underline{\mathbb{R}})$, we have $\alpha \in \mathrm{Im}(H^2(X, \underline{\mathbb{Z}}) \to H^2(X, \underline{\mathbb{R}}))$. Then we have the following result.

Theorem 2.10 (Lefschetz theorem on $(1,1)$-classes)**.** *Let X be a complex manifold, and let $\omega \in \mathcal{C}^\infty(X, \mathcal{A}^{1,1}_{X,\mathbb{R}})$ be a smooth, real d-closed $(1,1)$-form. Assume that $[\omega] \in H^2(X, \mathbb{R})$ is an integral class. Then there exists a hermitian line bundle (L, h) on X such that $\Theta_h(L) = \omega$. In particular, $[\omega] = c_1(L)$.*

Proof. As before, we use a cover by polydisks U_α such that the double overlaps $U_{\alpha\beta}$ are simply connected. One can write $\omega|_{U_\alpha} = dv_\alpha$ for some real 1-form $v_\alpha = v_\alpha^{1,0} + v_\alpha^{0,1}$, hence $\omega|_{U_\alpha} = \partial v_\alpha^{0,1} + \bar{\partial} v_\alpha^{1,0}$ and $\bar{\partial} v_\alpha^{0,1} = \omega|_{U_\alpha}^{0,2} = 0$ so that $v_\alpha^{0,1} = \bar{\partial} u_\alpha$ for some function u_α by Dolbeault-Grothendieck lemma. Set $\phi_\alpha := i(\bar{u}_\alpha - u_\alpha) \in \mathcal{C}^\infty(U_\alpha, \mathbb{R})$ so that

$$\omega|_{U_\alpha} = \frac{i}{2\pi}\partial\bar{\partial}\phi_\alpha$$

Define $d^c = \frac{i}{4\pi}(\bar{\partial} - \partial)$ so that $\omega|_{U_\alpha} = dd^c\phi_\alpha$. The class $[\omega]$ corresponds in $\check{H}^2(X, \underline{\mathbb{R}})$ to the class of the 2-cocycle $(f_{\alpha\beta\gamma})$ with real values defined by $f_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma}$ and where $f_{\alpha\beta}$ is any function such that $d^c(\phi_\alpha - \phi_b) = df_{\alpha\beta}$. By assumption, there exists an integral 2-cocyle $(n_{\alpha\beta\gamma})$ and a real 1-cycle $u = (u_{\alpha\beta})$ such that

$$f_{\alpha\beta\gamma} = n_{\alpha\beta\gamma} + (\delta u)_{\alpha\beta\gamma}$$

where δ is the Čech differential.

Now, on the overlap $U_{\alpha\beta}$, we have $\partial\bar{\partial}(\phi_\alpha - \phi_\beta) = 0$ so there exists a holomorphic function $g_{\alpha\beta} \in \mathcal{O}_X(U_{\alpha\beta})$ such that $2\mathrm{Re}(g_{\alpha\beta}) = \phi_\alpha - \phi_\beta$. This implies that $d^c(\phi_a - \phi_b) =$

$\mathrm{Re}(d^c g_{\alpha\beta}) = \frac{1}{\pi}\mathrm{Im}(dg_{\alpha\beta})$. By what was said above, there exists $u_{\alpha\beta}$ a real 1-cycle such that if one sets $g'_{\alpha\beta} := \frac{1}{\pi}g_{\alpha\beta} - iu_{\alpha\beta}$, then $\mathrm{Im}(g'_{\alpha\beta\gamma}) \in \mathbb{Z}$. Set $\widehat{g}_{\alpha\beta} := e^{\pi g'_{\alpha\beta}} = e^{g_{\alpha\beta}-\pi i u_{\alpha\beta}}$; it defines an element in $H^1(X, \mathcal{O}_X^*)$ (hence a line bundle L) thanks to that last integrality property. Moreover, the functions ϕ_α define a metric h on L (by the relation $|e_\alpha|_h^2 = e^{-\phi_\alpha}$) since $\phi_\alpha - \phi_\beta = 2\mathrm{Re}(g_{\alpha\beta}) = \log|\widehat{g}_{\alpha\beta}|^2$ and also, one has $\Theta_h(L) = dd^c\phi_\alpha = \omega$. □

Definition 2.11 (Néron-Severi group)**.** Let X be a compact Kähler manifold. The Néron-Severi group of X, denoted by $\mathrm{NS}(X)$, is defined as the image of the Chern class morphism

$$\mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{R}).$$

Thanks to Lefschetz theorem, we have $\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$.

The Néron-Severi group $\mathrm{NS}(X)$ is a finitely generated abelian subgroup of $H^2(X, \mathbb{R})$, hence torsion-free, of the form $\mathrm{NS}(X) = \bigoplus_{i\in I}\mathbb{Z}\alpha_i$ for some $\alpha_i = c_1(L_i)$. The rank of $\mathrm{NS}(X)$, i.e. $|I|$, is called the Picard rank of X and is usually denoted by $\rho(X)$. One has the obvious inequality $\rho(X) \leqslant h^{1,1}$ where $h^{1,1} = \dim_{\mathbb{C}} H^{1,1}(X) = \dim_{\mathbb{R}} H^{1,1}(X, \mathbb{R})$. We set $\mathrm{NS}(X)_{\mathbb{R}} := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ to be the vector space generated by $\mathrm{NS}(X)$; it has dimension $\rho(X)$.

Remark 2.12 (On the Picard rank)**.** The vector space $\mathrm{NS}(X)_{\mathbb{R}} \subset H^{1,1}(X, \mathbb{R})$ can be very small even though $h^{1,1}$ is large. For instance, a "general" torus $X = \mathbb{C}^n/\Lambda$ satisfies $\rho(X) = 0$, while $h^{1,1}(X) = \frac{n(n-1)}{2}$, cf Example 2.13. Such an example can of course never be a projective manifold (for which $\rho(X) \geqslant 1$, thanks to the existence of an ample line bundle arising as the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ under an embedding $X \hookrightarrow \mathbb{P}^N$).

The gap phenomenon $\rho < h^{1,1}$ may still occur if X is projective, but of course in that case one has $\rho \geqslant 1$. Examples can already be found dimension two, as there exist projective surfaces (called $K3$ surfaces) for which $\rho(X) = 1$ but $h^{1,1}(X) = 20$.

Example 2.13 (A Kähler manifold without proper submanifolds)**.** Let $X = \mathbb{C}^2/\Lambda$ be a complex 4-torus, where $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4$. We claim that for (e_i) "general", then X does not admit any proper submanifold $C \subset X$ and $\rho(X) = 0$. In particular, X is not projective (algebraic).

Arguing by contradiction, let C be a compact complex curve in X. Since X is diffeomorphic to the standard torus $\mathbb{C}^2/\mathbb{Z}^4$, we have $H_2(X, \mathbb{Z}) \simeq \mathbb{Z}^6$ with generators given by 6 cycles S_{ij}, the image in X of the planes $\mathbb{R}e_i + \mathbb{R}e_j$, $1 \leqslant i < j \leqslant 4$. Then one can decompose $[C] = \sum a_{ij}S_{ij}$ for $a_{ij} \in \mathbb{Z}$. In homology, one has $[C] \neq 0$ since $\int_C \omega > 0$ where ω is the Kähler form on X coming from the euclidean Kähler form $idz_1 \wedge d\bar{z}_1 + idz_2 \wedge d\bar{z}_2$ on $\mathbb{C}^2$ which is invariant under Λ.

Now, let η be the holomorphic 2-form on X coming from the Λ-invariant form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$. Write $e_i = (\alpha_i, \beta_i) \in \mathbb{C}^2$. The inclusion map $S_{ij} \hookrightarrow X$ can be read as a map $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}^2/\Lambda$ sending (u, v) to $(u\alpha_i + v\alpha_j, u\beta_i + v\beta_j)$ so that $f^*\eta = (\alpha_i\beta_j - \alpha_j\beta_i)du \wedge dv$ hence $\int_{S_{ij}} \eta = \alpha_i\beta_j - \alpha_j\beta_i$. Moreover, one has $\int_C \eta = 0$ for degree reason, so we get the relation $\sum_{i,j} a_{ij}(\alpha_i\beta_j - \alpha_j\beta_i) = 0$.

Now, if we choose the α_i, β_i in such a way that $\alpha_i\beta_j$ are linearly independent over $\mathbb{Z}$, we get a contradiction. It is easy to construct such numbers α_i, β_j explicitly. Alternatively, for each $(a_{ij}) \in \mathbb{Z}^6$, the equation $\sum_{i,j} a_{ij}(\alpha_i\beta_j - \alpha_j\beta_i) = 0$ defines a hypersurface in $H_a \subset \mathbb{C}^8$ and it is enough to choose $(\alpha, \beta) \in (\mathbb{C}^2)^4 \setminus \cup_{a\in\mathbb{Z}^6\setminus 0} H_a$ linearly independent over $\mathbb{C}$ (an open condition).

The claim on the Picard rank of X follows easily from the argument above. Indeed, we have showed more generally that any class $[c] \in H_2(X, \mathbb{Q})$ such that $\int_c \eta = 0$ must be zero. We claim that $H^2(X, \mathbb{Q}) \cap H^{1,1}(X) = 0$, which clearly implies that $\rho(X) = 0$. Indeed, if $[\alpha] \in H^2(X, \mathbb{Q}) \cap H^{1,1}(X)$, it induces a unique linear form $\Phi_\alpha \in H^2(X, \mathbb{Q})^\vee$ defined by $[\gamma] \mapsto \int_X \alpha \wedge \gamma$. By the universal coefficient theorem, the map $H_2(X, \mathbb{Q}) \to H^2(X, \mathbb{Q})^\vee$ sending $[c]$ to $[c]\cap$ is an isomorphism. Therefore, there exists $[c_\alpha] \in H^2(X, \mathbb{Q})$ such that $\int_X \alpha \wedge \gamma = \int_{c_\alpha} \gamma$ for all $\gamma \in H^2(X, \mathbb{Q})$. Applying this to $\gamma = \eta$, we find $\int_{c_\alpha} \eta = 0$, hence $[c_\alpha] = 0$ and $[\alpha] = 0$.

2.2 Divisors and line bundles

In this section X is a complex manifold of dimension n.

2.2.1 Divisors

Definition 2.14 (Subvariety)**.** An (analytic) subvariety of X is a closed subset $Y \subset X$ such that for any $x \in X$, there exists an open neighborhood $x \in U \subset X$ such that $Y \cap U$ is the zero set of finitely many holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}_X(U)$.

A point $x \in Y$ is said smooth or regular if one can choose the functions f_i such that the Jacobian of the holomorphic map $f = (f_1, \ldots, f_k)$ has rank k under a local trivialization. Otherwise, $x \in Y$ is singular.

Definition 2.15 (Dimension)**.** One can show that the set Y_{reg} of regular points of an analytic subvariety $Y \subset X$ is a non-empty complex submanifold of X. We define the dimension of Y by $\dim Y := \dim Y_{\text{reg}}$.

Definition 2.16 (Irreducible subvariety)**.** A subvariety $Y \subset X$ is said irreducible if it cannot be written as the union $Y = Y_1 \cup Y_2$ of two proper analytic subvarieties (i.e. $Y_1 \not\subset Y_2$ and $Y_2 \not\subset Y_1$)

Definition 2.17 (Hypersurface)**.** An hypersurface of X is a subvariety $H \subset X$ of codimension one, i.e. $\dim H = n - 1$.

Definition 2.18 (Divisors and $\mathbb{Q}$-divisors)**.** A divisor (resp $\mathbb{Q}$-divisor) is a formal linear combination $D = \sum_{i=1}^k a_i D_i$ where D_i is an irreducible hypersurface and $a_i \in \mathbb{Z}$ (resp. $a_i \in \mathbb{Q}$). We say that D is *effective* if each a_i is non-negative.

We will admit that similarly to the case of Riemann surfaces, given a meromorphic function $f \in \mathcal{M}_X(X)$, one can attach its divisor of zeros and poles $\text{div}(f) = \sum a_i D_i$ where f vanishes at order a_i along D_i if $a_i \geqslant 0$ and f has a pole of order $-a_i$ if $a_i \leqslant 0$.

Given the local nature of that construction, one can equally define a divisor $\text{div}(s)$ for any meromorphic section of a line bundle L on X.

Definition 2.19 (Equivalence of divisors)**.** We say that two divisors D, D' are linearly equivalent if there exists a meromorphic function $f \in \mathcal{M}_X(X)$ such that $D - D' = \text{div}(f)$. The group of isomorphism classes of divisors under linear equivalence is denoted by $\text{Div}(X)$.

2.2.2 Line bundle associated to a divisor

A smooth hypersurface $H \subset X$ is given locally on a collection of charts $U_\alpha \subset X$ as the zero locus $H \cap U_\alpha = (f_\alpha = 0)$ for some holomorphic function $f_\alpha \in \mathcal{O}_X(U_\alpha)$ such that ∇f_α never vanishes on U_α. It is easy to check that on the overlap $U_{\alpha\beta}$, we have $f_\alpha = g_{\alpha\beta} f_\beta$ for some non-vanishing holomorphic function $g_{\alpha\beta} \in \mathcal{O}_X(U_{\alpha\beta})^*$. Clearly, one has $g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$ on any triple overlap $U_{\alpha\beta\gamma}$. This allows one to define a cocycle $(g_{\alpha\beta}) \in Z^1(X, \mathcal{O}_X^*)$ and it is not hard to check that its class $[(g_{\alpha\beta})] \in H^1(X, \mathcal{O}_X^*)$ is independent of the choice of the functions f_α.

We will admit that the same construction can be carried over similarly if H is merely an analytic hypersurface.

Definition 2.20 (Line bundle associated to an hypersurface)**.** If $H \subset X$ is an analytic hypersurface, the cocycle $[(g_{\alpha\beta})] \in H^1(X, \mathcal{O}_X^*)$ constructed above yields a line bundle that we denote $\mathcal{O}_X(H)$.

Lemma 2.21. *Let $H \subset X$ be an hypersurface. The line bundle $\mathcal{O}_X(H)$ admits a holomorphic section s_H whose divisor is exactly H. It is unique up to an element of $\mathcal{O}_X(X)^*$.*

Proof. This is almost tautological, as we defined s_H by the data of the holomorphic function $\sigma_\alpha = f_\alpha$ on U_α, which satisfies $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$ automatically. As for uniqueness, if s, s' are two such sections, then $\frac{s}{s'}$ is a meromorphic function whose divisor of poles and zero is empty; i.e. it is an element of $\mathcal{O}_X(X)^*$. □

Definition 2.22 (Line bundle associated to a divisor)**.** Let $D = \sum_{i=1}^k a_i D_i$ be a divisor. We define $\mathcal{O}_X(D) := \mathcal{O}_X(D_1)^{\otimes a_1} \otimes \cdots \otimes \mathcal{O}_X(D_k)^{\otimes a_k}$.

In terms of cocycles, if $(f_\alpha^{(i)} = 0)$ is the equation of $D_i \cap U_\alpha$, then the cocycle associated to $\mathcal{O}_X(D)$ is simply $\prod_{i=1}^k \left(\frac{f_\alpha^{(i)}}{f_\beta^{(i)}}\right)^{a_i}$. Moreover, $\mathcal{O}_X(D)$ admits a meromorphic section $s_D := \prod_{i=1}^k s_{D_i}^{\otimes a_i}$ which satisfies $\mathrm{div}(s_D) = D$.

Lemma 2.23. *Let L be a line bundle on X endowed with a meromorphic section s. Then L is isomorphic to $\mathcal{O}_X(\mathrm{div}(s))$.*

Proof. The meromorphic section s correspond to meromorphic functions σ_α on U_α such that $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$ if $g_{\alpha\beta}$ are the transition functions of L. Set $D = \mathrm{div}(s)$. Since, locally, $D \cap U_\alpha = \mathrm{div}(\sigma_\alpha)$, the line bundle $\mathcal{O}_X(D)$ can be defined by the cocycle $\frac{\sigma_\alpha}{\sigma_\beta}$ which coincides with $g_{\alpha\beta}$. This proves the lemma. □

Lemma 2.24. *Let D be a divisor. Then $\mathcal{O}_X(D)$ is isomorphic to the trivial line bundle $\mathcal{O}_X$ if and only if $D = \mathrm{div}(f)$ for some meromorphic function $f \in \mathcal{M}_X(X)$.*

Proof. If $D = \mathrm{div}(f)$, then one chooses as local equation $(f = 0)$ on U_α and the cocycle $g_{\alpha\beta}$ is noting but $\frac{f}{f} \equiv 1$. Conversely assume that we have a trivializing section $s \in H^0(X, \mathcal{O}_X(D))$. Then s_D/s is a meromorphic function whose divisor is $\mathrm{div}(s_D) = D$. □

Corollary 2.25. *The map*

$$\begin{aligned} \mathrm{Div}(X) &\longrightarrow \mathrm{Pic}(X) \\ D &\longmapsto \mathcal{O}_X(D) \end{aligned}$$

is an injective morphism of abelian groups. Moreover, the image consists exactly of line bundles admitting a non-zero meromorphic section.

Proof. The fact that the map is well-defined and injective follows from Lemma 2.24. The description of the image follows from Lemmas 2.21-2.23. □

Remark 2.26. If X is a projective manifold, then the above map is surjective, hence isomorphic. Indeed, if L is a line bundle and H is an ample hypersurface, then for $p \gg 1$, $L \otimes H^{\otimes p}$ is globally generated, hence it has a non-zero section s. Then, $s/s_H^{\otimes p}$ is a non-zero rational section of L.

3 Positivity notions for $(1,1)$-classes

3.1 Positive and semipositive forms

Let X be a complex manifold of dimension n. As a complex manifold, it comes equipped with an orientation and one can choose a volume form dV (i.e. a smooth (n,n)-form which is nowhere zero). In particular, there is a notion of semipositivity for real $2n$-forms $\omega_x \in \Lambda^{2n}(T_{X,\mathbb{R}})_x$. Indeed, there exists $f_x \in \mathbb{R}$ such that $\omega_x = f_x dV$, and one says that ω_x is semipositive if $f_x \geqslant 0$.

Building on that definition, one gets a notion of semipositivity for real (p,p)-forms.

Definition 3.1 (Semipositive forms)**.** Let ω be a smooth, real (p,p)-form. One says that ω is semipositive if for any $x \in X$ and any $(1,0)$-forms $\alpha_1, \dots, \alpha_{n-p} \in (T^{1,0}_{X,x})^*$, one has $\omega_x \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_{n-p} \wedge \bar{\alpha}_{n-p} \geqslant 0$.

Remark 3.2 (Equivalent definitions)**.** The definition above is intrinsic (it does not depend on a choice of coordinates), and one can easily check that a real $(1,1)$-form ω is semipositive if and only if one of the following equivalent conditions holds

- for any $x \in X$ and any $u \in T_{X,x}$, we have $\omega_x(u, \bar{u}) \geqslant 0$.
- given a system of holomorphic coordinates $(z_1, \dots, z_n)$ where $\omega = i\sum_{k,\ell} \omega_{k\bar{\ell}} dz_k \wedge d\bar{z}_\ell$, the hermitian matrix $(\omega_{k\bar{\ell}})_{k\bar{\ell}}$ is semipositive .

If ω is closed, then it is locally dd^c-exact (recall that d^c is the real operator $\frac{i}{4\pi}(\bar{\partial} - \partial)$ so that $\frac{i}{2\pi}\partial\bar{\partial} = dd^c$), cf. beginning of proof of Theorem 2.10. On a chart where $\omega = dd^c\varphi$, then ω is semipositive iff φ is plurisubharmonic (psh).

Proposition 3.3. *Let $f : X \to Y$ be a holomorphic map between complex manifold and let ω be a semipositive (p,p)-form on Y. Then $f^*\omega$ is semipositive as well.*

Proof. This is a pointwise property, so one can assume that ω is a constant semipositive (p,p)-form on $\mathbb{C}^m$ and $f : \mathbb{C}^n \to \mathbb{C}^m$ is a complex linear map. Given $n-p$ independent $(1,0)$-forms $\alpha_1, \ldots, \alpha_{n-p}$ (which we complete to a basis of $(\mathbb{C}^n)^*$), let us define $S := \mathrm{Ker}\,(\alpha_1) \cap \ldots \cap \mathrm{Ker}\,(\alpha_{n-p})$ which is p-dimensional and satisfies

$$f^*\omega \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_{n-p} \wedge \bar{\alpha}_{n-p} = \lambda \bigwedge_{k=1}^{n} i\alpha_k \wedge \bar{\alpha}_k$$

where λ is defined by $(f^*\omega)|_S = \lambda \bigwedge_{k=n-p+1}^{n} i\alpha_k \wedge \bar{\alpha}_k$. Now, $(f^*\omega)|_S = f^*(\omega|_{f(S)})$ is zero unless $f|_S : S \to f(S)$ is isomorphic. Therefore, one can restrict to the case where $f : \mathbb{C}^p \to \mathbb{C}^p$ is a complex linear isomorphism and ω is a semipositive (p,p)-form on $\mathbb{C}^p$. We can write $\omega = \mu \bigwedge_{k=1}^{p} idz_k \wedge d\bar{z}_k$ with $\mu \geqslant 0$ by assumption and then get $f^*\omega = \mu|\det(f)|^2 \bigwedge_{k=1}^{p} idz_k \wedge d\bar{z}_k$. Therefore, $f^*\omega$ is semipositive as well. □

Proposition 3.4. *Let $\omega_1, \ldots, \omega_r$ be real, semipositive $(1,1)$-forms. Then, $\omega_1 \wedge \ldots \wedge \omega_r$ is semipositive.*

Proof. This is a pointwise property. Let $\omega = i\sum_{j,k} \omega_{jk} dz_j \wedge d\bar{z}_k$ be a semipositive $(1,1)$-form. Since $\Omega := (\omega_{jk})_{jk}$ is hermitian, there exists a unitary matrix P and a diagonal matrix D with non-negative coefficients such that $P\Omega P^* = D$. Set $dw_j := \sum_k p^{kj} dz_k$; it yields another basis of $(\mathbb{C}^n)^*$ and an easy computation shows that $\omega = i\sum_j \lambda_j dw_j \wedge d\bar{w}_j$.

We can perform this operation for $\omega_1, \ldots, \omega_r$ and we see that $\omega_1 \wedge \ldots \wedge \omega_r$ is a non-negative combination of (r,r)-forms of the form $i\alpha_1 \wedge \bar{\alpha}_1 \wedge \ldots \wedge i\alpha_r \wedge \bar{\alpha}_r$. In particular, it is semipositive. □

From the equivalent definitions above, it is clear that we can also define the notion of positive $(1,1)$-form (by replacing $\geqslant 0$/semipositive/psh by > 0/positive/strictly psh):

Definition 3.5 (Positive forms)**.** One says that a real $(1,1)$-form ω is positive if if one of the following equivalent conditions holds

- for any $x \in X$ and any $u \in T_{X,x} \setminus \{0\}$, we have $\omega_x(u, \bar{u}) > 0$.
- given a system of holomorphic coordinates $(z_1, \ldots, z_n)$ where $\omega = i\sum_{k,\ell} \omega_{k\bar{\ell}} dz_k \wedge d\bar{z}_\ell$, the hermitian matrix $(\omega_{k\bar{\ell}})_{k\bar{\ell}}$ is definite positive .

Definition 3.6 (Kähler form, II)**.** A Kähler form is a real, positive $(1,1)$-form ω which is closed, i.e. such that $d\omega = 0$.

3.2 Positive currents

Definition 3.7 (Positive current)**.** Let T be a smooth, real (p,p)-current. One says that T is positive if for any open set $U \subset X$ and any semipositive $(n-p, n-p)$-form ω with compact support on U, we have $\langle T, \omega \rangle \geqslant 0$.

If T is closed, then it is locally dd^c-exact and on a chart where $T = dd^c\varphi$, then T is positive iff φ is psh (to be rigourous, one should say that φ coincides almost everywhere with a psh function).

Given a system of holomorphic coordinates $(z_1, \ldots, z_n)$, one can uniquely write any (p,p)-current T as $T = i^{p^2} \sum T_{IJ} dz_I \wedge d\bar{z}_J$, where I, J run among the subsets of $\{1, \ldots n\}$ with cardinality p. Here, T_{IJ} is the distribution (or current of degree 0) defined up to a sign by $T_{IJ}(f) = \langle T, f i^{n-p} dz_{^cI} \wedge d\bar{z}_{^cJ} \rangle$. Recall that a distribution which takes non-negative values when evaluated against non-negative functions is automatically a positive measure, or equivalently, its extends as a continuous linear form on the space of compactly supported *continuous* functions.

Lemma 3.8. *If $T = i^{p^2} \sum T_{IJ} dz_I \wedge d\bar{z}_J$ is a positive current on $\mathbb{C}^n$, then the distributions T_{II} are positive measures, and T_{IJ} are complex measures. Moreover, one can check that the absolute values $|T_{IJ}|$ of the measures T_{IJ} are dominated by the mass of the measure $\sum T_{KK}$ where K ranges among all subsets contained in $I \cup J$ and containing $I \cap J$.*

Sketch of proof. In order to explain why, let us take a simple example $T = i \sum_{1 \leqslant j,k \leqslant 2} T_{jk} dz_j \wedge d\bar{z}_k$. Then for any smooth, compactly supported positive function $f \geqslant 0$, we have $T_{11}(f) = \langle T, f i dz_2 \wedge d\bar{z}_2 \rangle \geqslant 0$ hence T_{11} is a positive measure. Moreover, the identity

$$\begin{aligned} 4 dz_1 \wedge d\bar{z}_2 =& (dz_1 + dz_2) \wedge \overline{(dz_1 + dz_2)} - (dz_1 - dz_2) \wedge \overline{(dz_1 - dz_2)} + \\ &+ i(dz_1 + i dz_2) \wedge \overline{(dz_1 - i dz_2)} - i(dz_1 - i dz_2) \wedge \overline{(dz_1 + i dz_2)} \end{aligned}$$

shows that T_{21} is a complex linear combination of (four) positive measures of the form $f \mapsto \langle T, f i \alpha \wedge \bar{\alpha} \rangle$ where α is a $(1,0)$-form satisfying $i\alpha \wedge \bar{\alpha} \leqslant \frac{1}{2}(i dz_1 \wedge d\bar{z}_1 + i dz_2 \wedge d\bar{z}_2)$. In particular, the absolute value of T_{21} is less than $2\langle T, i dz_1 \wedge d\bar{z}_1 + i dz_2 \wedge d\bar{z}_2 \rangle = 2(\|T_{11}\| + \|T_{22}\|)$. □

If $\omega_{\mathbb{C}^n} := i \sum_{k=1}^n dz_k \wedge d\bar{z}_k$ is the standard euclidean form on $\mathbb{C}^n$, the trace measure of a real, positive (p,p)-current T with respect to ω is defined by $T \wedge \omega_{\mathbb{C}^n}^{n-p}$. This is a positive measure, which coincides up to a (positive) coefficient with $\sum_I T_{II}$ where $I \subset \{1, \ldots, n\}$ ranges among all subsets with cardinality p.

Proposition 3.9 (Weak compactness for positive currents)**.** *Let X be a compact complex manifold of dimension n, and let ω be a positive $(1,1)$-form. For any $C > 0$, the set of positive (p,p)-currents T such that $\int_X T \wedge \omega^{n-p} \leqslant C$ is weakly compact.*

In particular, if X is Kähler and $\alpha \in H^2(X, \mathbb{R})$ is fixed, then the set of closed, positive $(1,1)$-currents $T \in \alpha$ is weakly compact. More generally, if $\mathcal{C} \subset H^2(X, \mathbb{R})$ is a compact subset, then the set of closed, positive $(1,1)$-currents $T \in \mathcal{C}$ is weakly compact.

Proof. The second assertion follows immediately from the first one since if ω is Kähler and $T \in \alpha$, then $\int_X T \wedge \omega^{n-1} = \alpha \cup [\omega]^{n-1}$ is independent of T.

Let us now prove the first assertion. It is enough to work locally on a given trivializing chart U and use a diagonal extraction. Then one can find a constant $A > 0$ such that $\omega \leqslant A\omega_{\mathbb{C}^n}$, and we get that $\int_U T \wedge \omega_{\mathbb{C}^n}^{n-1} \leqslant A^{n-1} C$. In particular, the positive measure T_{II} have mass bounded independently of T, and the absolute values of T_{IJ} are bounded as well by Lemma 3.8 above. Given the compactness of complex measures with absolute values bounded by a constant (for the weak topology), the result follows.

As for the last point, if ω is a Kähler metric, then the formula $\int_X T \wedge \omega^{n-1} = [T] \cup [\omega]^{n-1}$ obtained from Lemma 1.9 shows that the quantity only depends on $\mathcal{C}$ and not

T. This shows one can extract convergent subsequences of any families of such currents T, and one only has to check that the limit belong to $\mathcal{C}$. But this follows from the fact that the "cohomology map" $T \mapsto [T] \in H^2(X, \mathbb{R})$ is continuous for the weak topology, cf Proposition 1.10. $\square$

Definition 3.10 (Current of integration, II)**.** Let X be a complex manifold of dimension n, and let $Z \subset X$ be a closed k-dimensional complex submanifold. The current of integration $[Z]$ is the current of degree $(n-k, n-k)$ defined by

$$\langle [Z], \alpha \rangle := \int_Z \alpha$$

where $\alpha \in \mathcal{C}_c^\infty(X, \mathcal{A}_X^{k,k})$ and the integral on the RHS is defined to be $\int_Z j^*\alpha$ if $j : Z \hookrightarrow X$ is the closed immersion. The current $[Z]$ is closed and positive.

Definition 3.11 (Order relation)**.** Given two currents real (p,p)-currents T, T', we say that $T \geqslant T'$ if $T - T'$ is a positive current.

Note that if ω is a smooth semipositive (p,p)-form, then it induces a positive (p,p)-current. Conversely, if a smooth real (p,p)-form is positive (seen as a current), then it is semipositive (as a smooth form).

Definition 3.12 (Kähler currents)**.** One says that a real $(1,1)$-current T is a Kähler current if there exists a positive form ω such that $T \geqslant \omega$.

3.3 Quasi-plurisubharmonic functions

Definition 3.13 (Plurisubharmonic function)**.** A plurisubharmonic function (psh for short) on a connected open set $\Omega \subset \mathbb{C}^n$ is a function $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ such that

- φ is upper semi-continuous
- For all $a \in \Omega, \xi \in \mathbb{C}^n$ with $|\xi| = 1$, and $r > 0$ such that $\bar{B}(a,r) \subset \Omega$,

$$\varphi(a) \leqslant \int \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + re^{i\theta}\xi) d\theta.$$

If $f : \Omega' \to \Omega$ is a holomorphic map between domains in the complex euclidean space and φ is a psh function on Ω, then $f^*\varphi$ is a psh function on Ω' (possibly $\equiv -\infty$). In particular, the notion of psh function is well-defined on a complex manifold.

Definition 3.14 (Quasi-plurisubharmonic function)**.** Let X be a complex manifold, let θ be a smooth $(1,1)$-form and let $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ be a function. One says that

- φ is quasi-plurisubharmonic (quasi-psh for short) if φ is locally the sum of a psh function and a smooth function. In other words, for any $x \in X$, there exist a neighborhood $U \in x$, a psh function ψ on U, a smooth function f on U such that $\varphi|_U = \psi + f$.
- φ is θ-psh if φ is quasi-psh and $\theta + dd^c\varphi \geqslant 0$.

Let us recall the following important result coming from Hodge theory.

Lemma 3.15 ($\partial\bar{\partial}$-lemma)**.** *Let X be a compact Kähler manifold, and let α be two real, closed $(1,1)$-forms such that $[\alpha] = 0 \in H^2(X, \mathbb{R})$. Then, there exists $f \in \mathcal{C}^\infty(X, \mathbb{R})$ such that $\alpha = \partial\bar{\partial} f$.*

The same result holds if α is merely a closed $(1,1)$-current such that $[\alpha] = 0 \in H^2_{cur}(X, \mathbb{R})$, but then the potential f has to belong to $L^1(X)$.

In particular, if X is a compact Kähler manifold and T is a closed $(1,1)$-current, then one can write $T = \theta + dd^c\varphi$ for some (unique modulo constants) function $\varphi \in L^1(X)$, and some smooth closed $(1,1)$-form θ.

One says that T is quasi positive if there exists a $(1,1)$-form γ such that $T \geqslant \gamma$. In particular, if T is quasi-positive, then one can write $T = \theta + dd^c\varphi$ for some quasi-psh function φ. This allow to pull back T as follows:

Proposition 3.16 (Pull-back of a quasi-positive current)**.** *Let $f : X \to Y$ be a surjective holomorphic map between compact Kähler manifolds, and let T be a closed $(1,1)$-current on X such that $T \geqslant \gamma$ for some smooth $(1,1)$-form γ. Let $\alpha = [T] \in H^2(X, \mathbb{R})$ be its de Rham cohomology class. Then, there exists a $(1,1)$ current f^*T such that*

- *f^*T coincides with the usual pull-back if T is a smooth $(1,1)$-form.*
- *f^*T is closed and $[f^*T] = f^*\alpha \in H^2(X, \mathbb{R})$.*
- *$f^*T \geqslant f^*\gamma$; in particular, it is quasi-positive.*

Proof. The construction is straightforward: write $T = \theta + dd^c\varphi$. Then φ is quasi-psh, hence $f^*\varphi$ is quasi-psh as well and not identically $-\infty$ since f is surjective. We then set $f^*T = f^*\theta + dd^c f^*\varphi$ and it clearly satisfies the properties above. □

Proposition 3.17. *Let $f : X \to Y$ be a surjective holomorphic map with connected fibers between compact, complex manifolds. Let θ be a $(1,1)$-form on Y. Then, any $f^*\theta$-psh function ψ descends, i.e. there exists φ a θ-psh function on Y such that $\psi = f^*\varphi$.*

If X, Y are Kähler, then any closed, positive $(1,1)$-current $T \in [f^*\theta]$ can be written as $T = f^*S$ for some closed current $S \in [\theta]$.

Remark 3.18. By a theorem of Zariski, any bimeromorphic map between complex connected manifolds has connected fibers.

Proof. Let $y \in Y$ be a point, and let $F := f^{-1}(y)$. It is a connected, compact complex space. Moreover, $\psi|_F$ is $(f^*\theta)|_F$-psh, i.e. it is psh. By the maximum principle, $\psi|_F$ is constant. Therefore there exists a unique function φ on X such that $\psi = f^*\varphi$. We need to show that φ is θ-psh. The problem is local on Y, and after substracting a local potential of θ, we are reduced to showing that $f^*\varphi$ psh implies φ psh.

Let $Y^\circ \subset Y$ be the regular locus of f (i.e. the locus of point $y \in Y$ such that f has maximal rank at any point of $f^{-1}(y)$. The complement of Y° is an analytic subset of dimension strictly less that $\dim Y$. If $y \in Y^\circ$ and $x \in f^{-1}(y)$, there exists a chart U (resp. V) of $y \in Y$ (resp. $x \in X$) such that $f|_V : V \to U$ is given by $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_k)$. By assumption, our psh function $\psi = \psi(z_1, \ldots, z_n)$ satisfies $\psi(z_1, \ldots, z_n) = \varphi(z_1, \ldots, z_k)$ for some function φ. By restriction, $\psi(z_1, \ldots, z_k, 0, \ldots, 0) = \varphi(z_1, \ldots, z_k)$ is also psh.

To finish the proof, recall that a psh function u defined on the complement $\Omega \setminus A$ of an analytic subset A extends to a psh function u^* on Ω if and only if it is locally bounded above. Moreover, the psh extension u^* of u is unique, given by $u^*(a) = \limsup_{x\to a, x\notin A} u(x)$. In our situation, we therefore need to show that for any $y_0 \in Y \setminus Y^\circ$, we have $\varphi(y_0) = \limsup_{y\to y_0} \varphi(y)$. Since this is the case for $f^*\varphi$ and f is proper, we immediately get the equality. □

3.4 Positive classes

Let (X, ω) be a compact Kähler manifold, and let $\alpha \in H^{1,1}(X, \mathbb{R})$. Recall that α can be represented either by $\bar\partial$-closed smooth $(1,1)$-forms of by $\bar\partial$-closed $(1,1)$-currents.

Definition 3.19. We say that α is

- a *Kähler class* if there exists a Kähler form $\theta \in \alpha$.
- a *semipositive class* if there exists a semipositive form $\theta \in \alpha$.
- a *nef class* if for any $\varepsilon > 0$, there exists $\theta_\varepsilon \in \alpha$ such that $\theta_\varepsilon \geqslant -\varepsilon\omega$.
- a *big class* if there exists a Kähler current $T \in \alpha$.
- a *pseudoeffective class* (psef for short) if there exists a positive current $T \in \alpha$.

Definition 3.20. A line bundle $L \to X$ is said to be positive (resp. semipositive, nef, big, pseudoeffective) if $c_1(L)$ is a Kähler class (resp semipositive, nef class, big class, pseudoeffective class).

By Lemma 2.9, we see that L is

- positive iff it admits a smooth hermitian metric h such that $\Theta_h(L)$ is Kähler.
- semipositive iff it admits a smooth hermitian metric h such that $\Theta_h(L) \geqslant 0$.
- nef iff for any $\varepsilon > 0$, it admits a smooth hermitian metric h_ε such that $\Theta_{h_\varepsilon}(L) \geqslant -\varepsilon\omega$.
- big iff it admits a singular hermitian metric h such that $\Theta_h(L)$ is a Kähler current.
- pseudoeffective iff it admits a singular hermitian metric h with $\Theta_h(L) \geqslant 0$.

Lemma 3.21. *We have the implications*

Kähler ⇒ big ⇒ psef, and *Kähler ⇒ semipositive ⇒ nef ⇒ psef.*

Remark 3.22. As we will see later, the reverse implications do not hold in general.

Proof. The only non-trivial implication is the last one, and is a consequence of the compactness result given by Proposition 3.9. □

Definition 3.23 (Cones)**.** Let $\mathcal{C} \subset H^{1,1}(X, \mathbb{R})$. We say that $\mathcal{C}$ is a cone if for all $\alpha, \beta \in \mathcal{C}$, one has

- $\forall t > 0,\ t\alpha \in \mathcal{C}$.

- $\alpha + \beta \in \mathcal{C}$.

In particular, $\mathcal{C}$ is convex, i.e. it is stable under convex linear combination.

The vector space $H^{1,1}(X, \mathbb{R})$ is finite-dimensional, hence all norms are equivalent and we will fix one from now on. If $\mathcal{C} \subset H^{1,1}(X, \mathbb{R})$ is cone, then we say that $\mathcal{C}$ is a closed (resp. open) cone if it is closed (resp. open) for the natural topology in $H^{1,1}(X, \mathbb{R})$. A closed cone always contains 0.

We say that a cone $\mathcal{C}$ is a positive cone if $\forall \alpha \in H^{1,1}(X, \mathbb{R}), \alpha, -\alpha \in \mathcal{C} \Rightarrow \alpha = 0$.

Definition 3.24 (Positivity cones)**.** Let X be a compact Kähler manifold. We define

- the *Kähler cone* $\mathcal{K}$ to be the set of Kähler classes.
- the *nef cone* to be the set of nef classes.
- the *pseudoeffective cone* $\mathcal{E}$ to be the set of pseudoeffective classes.
- the *big cone* to be the set of big classes.

Proposition 3.25. *The cones defined above have the following property:*

1. *The Kähler cone $\mathcal{K}$ is open.*
2. *The nef cone coincides with the closure $\overline{\mathcal{K}}$ of the Kähler cone; in particular, it is closed.*
3. *The pseudoeffective cone $\mathcal{E}$ is closed.*
4. *The big cones coincides with the interior $\overset{\circ}{\mathcal{E}}$ of the pseudoeffective cone; it particular, it is open. Moreover, all the cones above are positive cones.*

Proof. The first item is clear.

For the second one, let α be a nef class and let θ_ε as in the definition. Then, $\alpha = \lim_{\varepsilon \to 0}(\alpha + 2\varepsilon[\omega])$ and the latter class is Kähler since it contains the Kähler form $\theta_\varepsilon + 2\varepsilon\omega$. In particular, the nef cone is contained in the closure of the Kähler cone. Conversely, if $\alpha = \lim \alpha_\delta$ is a limit of Kähler classes $\alpha_\delta = [\omega_\delta]$, then for any $\varepsilon > 0$, we have $\alpha \geqslant \alpha_\delta - \varepsilon[\omega]$ for $\delta \leqslant \delta(\varepsilon)$, i.e. the difference contains a semipositive form γ_δ. In particular $\theta_\varepsilon := \gamma_{\delta(\varepsilon)} + \omega_{\delta(\varepsilon)} - \varepsilon\omega \in \alpha$ is a smooth form such that $\theta_\varepsilon \geqslant -\varepsilon\omega$, hence α is nef.

The closedness of the psef cone is a consequence of Proposition 3.9.

The openness of the big cone is clear. It remains to show that it contains the interior of $\mathcal{E}$. It $\alpha \in \overset{\circ}{\mathcal{E}}$, then $\alpha - \varepsilon[\omega] \in \mathcal{E}$ for $\varepsilon \ll 1$. Let $T \geqslant 0$ be a positive current in the class $\alpha - \varepsilon[\omega]$. Then the current $T + \varepsilon\omega$ is Kähler and belongs to α, hence α is big.

As for the last property, since $\mathcal{E}$ contains all the other cones, it is sufficient to treat that case. Now, if $\alpha, -\alpha \in \mathcal{E}$, then we can find a current $T \geqslant 0$ (resp. $S \geqslant 0$) such that $T, -S \in \alpha$. Therefore, we have $\alpha \cdot [\omega]^{n-1} = \int_X T \wedge \omega^{n-1} = -\int_X S \wedge \omega^{n-1}$ which is both non-negative and non-positive, hence it is zero. Since $T \wedge \omega^{n-1}$ is the trace measure (it coincides with $\sum_I T_{II}$ in local coordinates, and dominates the absolute value of all other components of T), we infer that $T \equiv 0$ hence $\alpha = 0$. □

Lemma 3.26. *Let $\alpha_1, \dots, \alpha_n$ be Kähler (resp. nef) classes on X. Then the intersection product $\alpha_1 \cdot \dots \cdot \alpha_n$ is positive (resp. semipositive).*
More generally, if $\alpha_1, \dots, \alpha_{n-1}$ are nef and α_n is psef, then $\alpha_1 \cdot \dots \cdot \alpha_n \geqslant 0$.

Proof. By continuity of the intersection product and since a nef class is a limit of Kähler classes, one can assume that the α_i are Kähler. But then, if $\omega_i \in \alpha_i$ is Kähler, one has

$$\alpha_1 \cdot \dots \cdot \alpha_n = \int_X \omega_1 \wedge \dots \wedge \omega_n > 0.$$

One way to see this is to use the fact that a wedge product of semipositive forms remains semipositive (Proposition 3.4). Then one picks a Kähler form ω scaled so that $\omega_i \geqslant \omega$, and the integrand becomes larger that ω^n which is clearly strictly positive.

As for the last statement, let $T \in \alpha_n$ be a positive current. Then $T \wedge \omega_1 \wedge \cdots \omega_{n-1}$ is a positive current of degree (n,n), hence it has non-negative mass. Lemma 1.9 concludes the proof. □

Remark 3.27. The result is false if one replaces "nef" by "psef" or even "big", cf section 4.5.

Lemma 3.28. *We have $0 \in \partial\mathcal{K}$ and $0 \in \partial\mathcal{E}$.*

Proof. The fact that 0 lies in the closure of the Kähler cone (hence on the closure of the big cone) follows from $0 = \lim_{\varepsilon \to 0} \varepsilon[\omega]$ for any Kähler form ω. However, 0 is neither a Kähler not a big class. Indeed, if α is a big class, it contains a Kähler current $T \geqslant \omega$ for some Kähler form ω, hence $[\alpha] \cdot [\omega]^{n-1} = \int_X T \wedge \omega^{n-1} \geqslant \int_X \omega^n > 0$ by Lemma 1.9. □

3.5 First examples

3.5.1 The projective space

Let $\mathbb{P}^n$ be the space of lines through the origin in $\mathbb{C}^{n+1}$ and let $L := \mathcal{O}_{\mathbb{P}^n}(1)$ be the dual of the tautogical bundle $\mathcal{O}_{\mathbb{P}^n}(-1) = \{(x,v) \in \mathbb{P}^n \times \mathbb{C}^{n+1}; v \in x\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$. In other words, if $x = [z_0 : \dots : z_n] \in \mathbb{P}^n$, then the fiber L_x consists of all linear forms on the line $\mathbb{C}(z_0, \dots, z_n)$.

One can cover X by the charts $U_i = \{[z_0 : \dots : z_n] \in \mathbb{P}^n; z_i \neq 0\}$ which are isomorphic to $\mathbb{C}^n$ via the coordinates $w_0 = \frac{z_0}{z_i}, \dots, \widehat{w}_i, \dots, w_n = \frac{z_n}{z_i}$. Each linear projection $(\lambda z_0, \dots, \lambda z_i) \mapsto \lambda z_i$ defines a section e_i of L which is non-zero exactly on U_i, and one has $g_{ij} = \frac{e_j}{e_i} = \frac{z_j}{z_i}$ on U_{ij}.

If we set $\phi_i := -\log \frac{|z_i|^2}{\|z\|^2}$ on U_i, then $\phi_i - \phi_j = \log |g_{ij}|^2$ on U_{ij} hence $e^{-\phi_i}$ defines an hermitian metric h_{FS} on L. Its curvature can be computed on U_k using the w-coordinate as

$$\Theta_{h_{\mathrm{FS}}}(L)|_{U_i} = \frac{i}{2\pi} \partial\bar{\partial} \log(1 + \|w\|^2).$$

One has $\Theta_{h_{\mathrm{FS}}}(L)|_{U_i} = \frac{1}{2\pi} \frac{1}{(1+\|w\|^2)^2} \sum_{j,k} a_{jk} i dz_j \wedge d\bar{z}_k$ where $a_{jk} = (1 + \|w\|^2)\delta_{jk} - w_k \bar{w}_j$. The hermitian matrix (a_{jk}) has eigenvalues $(1 + \|w\|^2)$ with multiplicity $(n-1)$ and 1 with multiplicity 1, hence it is definite positive. Hence

$$\omega_{\mathrm{FS}} := \Theta_{h_{\mathrm{FS}}}(L)$$

is a Kähler form, called Fubini-Study metric, and $\mathcal{O}_{\mathbb{P}^n}(1)$ is positive.

Alternatively, one can describe h_{FS} more intrinsically as the dual metric of the restriction to $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ of the euclidean metric on $\mathbb{C}^{n+1}$. That is, if $x \in \mathbb{P}^n$ and $\phi \in \mathcal{O}_{\mathbb{P}^n}(1)_x = x^\vee$,

$$\|\phi\|^2_{h_{\mathrm{FS}}} = \sup\{|\phi(v)|^2 \,|\, v \in x, \|v\|^2 = 1\}.$$

If $x = [z_0 : \ldots : z_n] \in U_i$, we recover $\|e_i(x)\|^2_{h_{\mathrm{FS}}} = \sup\{|\lambda z_i|^2 | \sum_k |\lambda z_k|^2 = 1\} = \frac{|z_i|^2}{\sum_k |z_k|^2}$.

Since $H^2(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$ is generated by $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$, we have that every class $\alpha \in H^{1,1}(X, \mathbb{R})$ is either zero, Kähler, or anti-Kähler.

Remark 3.29. Let A be a definite positive hermitian matrix of size $n+1$, and let h_A be the dual of the hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ induced by restriction (e.g. $h_{I_{n+1}} = h_{\mathrm{FS}}$). The square root of A induces an isomorphism $f \in \mathrm{Aut}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1))$ such that $f^* h_{\mathrm{FS}} = h_A$. In particular, $\Theta_{h_A}(\mathcal{O}_{\mathbb{P}^n}(1)) = f^*\omega_{\mathrm{FS}}$ is definite positive.

3.5.2 Projective manifolds

Subvarieties.
Let $X \subset \mathbb{P}^n$ be a smooth complex submanifold of the projective space. The hermitian line bundle $(L, h) := (\mathcal{O}_{\mathbb{P}^n}(1)|_X, h_{\mathrm{FS}}|_X)$ defines a hermitian line bundle on X whose Chern curvature form is $\Theta_{h_{\mathrm{FS}}}(O_{\mathbb{P}^n}(1))|_X = \omega_{\mathrm{FS}}|_X$ so it is still a Kähler form and L is positive. In general though, the space $H^2(X, \mathbb{Z})$ might be much larger than the one for $\mathbb{P}^n$, cf Remark 2.12.

Products.
Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let $p : X \to \mathbb{P}^1$ be the projection to the first factor. Let $(L, h) := (p^*\mathcal{O}_{\mathbb{P}^1}(1), h_{\mathrm{FS}}|_X)$. Clearly, the curvature of (L, h) is $p^*\omega_{\mathrm{FS}}$ hence it is a semipositive smooth form. In particular, L is nef.

However, L is not positive. Indeed, let $\alpha = c_1(L) \in H^{1,1}(X, \mathbb{R})$. If L were positive, α would contain a Kähler metric ω and one would have $\alpha^2 = \int_X \omega^2 > 0$, but actually, one has $\alpha^2 = \int_X (p^*\omega_{\mathrm{FS}})^2 = \int_X p^*(\omega_{\mathrm{FS}}^2) = 0$ since $\dim \mathbb{P}^1 = 1$.

We claim that more generally, L is not big. Argue by contradiction and assume that α contains a Kähler current $T = \theta + dd^c\varphi$ where $\theta \in \alpha$ is smooth. Since $\varphi \not\equiv -\infty$, there exists $z \in \mathbb{P}^1$ such that $\varphi|_{p^{-1}(z)} \not\equiv -\infty$. Set $X_z := p^{-1}(z)$. In particular, it makes sense to restrict $T|X_z = \theta_{|X_z} + dd^c\varphi|_{X_z}$ which is a Kähler current living in the cohomology class of $L|_{X_z}$. Now $L|_{X_z} = (p^*O_{\mathbb{P}^1}(1))|_{X_z} = p|^*_{X_z} O_{\mathbb{P}^1}(1)|_z$ which is trivial, in contradiction with the fact that $c_1(L|_{X_z})$ is big.

Note that one could also have relied on Proposition 3.17 which would have yielded a current S on $\mathbb{P}^1$ such that $p^*S \geqslant \omega$ for some Kähler form ω on X. If ω_{FS} is the Fubini-Study metric on $\mathbb{P}^1$, we have $S \wedge \omega_{\mathrm{FS}} \equiv 0$ for degree reasons, hence $0 = p^*S \wedge p^*\omega_{\mathrm{FS}} \geqslant \omega \wedge p^*\omega_{\mathrm{FS}}$. But the integral of the RHS is strictly positive (the RHS is the trace measure of the non-zero form semipositive form $p^*\omega_{\mathrm{FS}}$), hence the contradiction.

3.5.3 Effective line bundles

Let X be a compact complex manifold and let $L \to X$ be a line bundle. We pick an arbitrary hermitian metric h on L, with curvature form $\Theta_h(L)$. In this section, we assume that L admits a non-zero section s, with zero divisor D. Recall from Lemma 2.23 that L is isomorphic to $\mathcal{O}_X(D)$.

Let $x \in X \setminus \mathrm{Supp}(D)$, and let $e \in L_x$. The fraction e/s is a well-defined complex number, and we define a metric by setting $\|e\|^2 := \left|\frac{e}{s}\right|^2$. Alternatively, one can artificially use h and set $\|e\|^2 := \frac{\|e\|_h^2}{\|s\|_h^2}$. This defines a singular metric h_D on L (or equivalently on $\mathcal{O}_X(D)$), which is smooth away from $\mathrm{Supp}(D)$.

Let us state a few easy properties satisfied by h_D. First, one has $\|s\|_{h_D}^2 \equiv 1$. Next, recall that the section s of $\mathcal{O}_X(D)$ corresponds in the charts U_α to the holomorphic function f_α defining D. A trivialization e_α of that line bundle over U_α corresponds to the function 1, hence $s = f_\alpha e_\alpha$ on U_α, and $\|e_\alpha\|_{h_D}^2 = \frac{1}{|f_\alpha|^2}$. In particular, the local weights are $\phi_\alpha = \log|f_\alpha|^2$. Thanks to Lelong-Poincaré formula, we get

$$\Theta_{h_D}(L) = [D]. \tag{3.11}$$

In particular, (L, h_D) is a singular hermitian line bundle with positive curvature, and L is pseudoeffective:

Corollary 3.30. *Let X be a compact Kähler manifold and let $D = \sum a_i D_i$ be an effective divisor (i.e. $(a_i \geqslant 0)$ for all i). Then $c_1(D)$ is a pseudoeffective class.*

Another way to rephrase (3.11) involving the background smooth metric h on L is to write on U_α the identity $\|s\|_h^2 = |f_\alpha|^2\|e_\alpha\|_h^2 = |f_\alpha|^2 e^{-\phi_\alpha^h}$ where ϕ_α^h is the local weight of h in U_α under the chosen trivialization. Taking the log and differentiating, we get

$$dd^c \log\|s\|_h^2 = [D] - \Theta_h(L). \tag{3.12}$$

In passing, we get the following corollary

Corollary 3.31. *Let X be a compact Kähler manifold, let H be a smooth hypersurface, and let $\alpha_1, \ldots, \alpha_{n-1} \in H^{1,1}(X, \mathbb{R})$. Then, we have*

$$\alpha_1 \cdot \ldots \cdot \alpha_{n-1} \cdot c_1(H) = \alpha_1|_H \cdot \ldots \cdot \alpha_{n-1}|_H.$$

When no confusion is possible, we will not distinguish between H and $c_1(H)$, and we will write $\alpha_1 \cdot \ldots \cdot \alpha_{n-1} \cdot H$ for $\alpha_1 \cdot \ldots \cdot \alpha_{n-1} \cdot c_1(H)$.

Proof. Let $\theta_i \in \alpha_i$ be a smooth representative. By (3.12), the current of integration along the hypersurface H, $[H]$, belong to the cohomology class $c_1(H)$. In particular, it follows from Lemma 1.9 that

$$\begin{aligned} c_1(H) \cdot \alpha_1 \cdot \ldots \cdot \alpha_{n-1} &= \int_X [H] \wedge \theta_1 \wedge \ldots \wedge \theta_{n-1} \\ &= \int_H \theta_1|_H \wedge \ldots \wedge \theta_{n-1}|_H \\ &= \alpha_1|_H \cdot \ldots \cdot \alpha_{n-1}|_H. \end{aligned}$$

□

One can make a similar construction given several sections $s_1, \dots, s_N$ of L by setting $\|e\|^2 := \frac{\|e\|_h^2}{\|s_1\|_h^2 + \dots + \|s_N\|_h^2}$, which is again independent of h. Let us denote this new metric by $\widetilde{h}_D$. In local charts U_α trivializing L where s_i correspond to a holomorphic function $f_{\alpha,i}$, the weight ϕ_α of $\widetilde{h}_D$ is $\log\left(\sum_{i=1}^N |f_{\alpha,i}|^2\right)$. In particular, the weights are psh and $\Theta_{\widetilde{h}_D}(L)$ is a positive current. Moreover, provided that the common zero locus $(s_1 = \dots = s_N = 0) = \emptyset$ is empty, then $\Theta_{\widetilde{h}_D}(L)$ is a *smooth* semipositive form.

4 Blow-ups

4.1 Normal bundle

Let X be a complex manifold and let $Y \subset X$ be a smooth submanifold of codimension $r \geqslant 1$.

Definition 4.1 (Normal bundle)**.** The normal bundle $N_{Y|X}$ of $Y \subset X$ is defined as the cokernel of the map $T_Y \longrightarrow T_X|_Y$, i.e.

$$0 \longrightarrow T_Y \longrightarrow T_X|_Y \longrightarrow N_{Y|X} \longrightarrow 0.$$

The normal bundle $N_{Y|X}$ is a vector bundle on Y with rank $r = \mathrm{codim}_Y X$.

When no confusion is possible, we sometimes write N_Y instead of $N_{Y|X}$.

Proposition 4.2. *Let $Y \subset X$ be a smooth hypersurface. Then $N_{Y|X} \simeq \mathcal{O}_Y(Y) := \mathcal{O}_X(Y)|_Y$.*

Proof. Cover X with charts U_α where $Y \cap U_\alpha = (f_\alpha = 0)$. On $U_{\alpha\beta}$, there exists a non-vanishing function h such that $f_\alpha = hf_\beta$ (h is nothing but the transition function of $\mathcal{O}_X(Y)$ relative to (U_α)). We thus have $df_\alpha = hdf_\beta + f_\beta dh$, hence $df_\alpha = hdf_\beta$ on $Y \cap U_{\alpha\beta}$.

By dualizing the exact sequence defining the normal bundle, we get

$$0 \longrightarrow N_Y^* \longrightarrow \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0.$$

The kernel N_Y^* is of rank one, locally generated on $Y \cap U_\alpha$ by the differential $e_\alpha := df_\alpha|_Y$. This means that the cocycle $(g_{\alpha\beta} = \frac{e_\beta}{e_\alpha})_{\alpha\beta}$ associated to N_Y^* is nothing but $\frac{df_\beta}{df_\alpha}|_Y = \frac{f_\beta}{f_\alpha}|_Y = h^{-1}$, the cocyle associated to $\mathcal{O}_X(-Y)|_Y$. This proves the formula. □

4.2 Projectivized bundles

Definition 4.3 (Projectivized bundle)**.** Let X be a complex manifold, and let $E \to X$ be a holomorphic vector bundle of rank $r \geqslant 2$. There exists a locally trivial fiber bundle $\mathbb{P}(E) \to X$ such that $\mathbb{P}(E)_x = \mathbb{P}(E_x) \simeq \mathbb{P}^{r-1}$ is the space of complex lines of E.

Example 4.4 (Projectivized normal bundle)**.** Let $Y \subset X$ be a smooth submanifold of codimension $r \geqslant 2$. The projectivized normal bundle, $\mathbb{P}(N_{Y|X})$, is the bundle of normal directions. It fibers over Y with fibers isomorphic to $\mathbb{P}^{r-1}$.

Definition 4.5 (Tautological bundle)**.** Let $E \to X$ be a holomorphic vector bundle of rank $r \geqslant 2$, inducing $\pi : \mathbb{P}(E) \to Y$. The tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathbb{P}(E)$ is defined by $\mathcal{O}_{\mathbb{P}(E)}(-1) := \{((x,[v]),w) \in \mathbb{P}(E) \times E_x | w \in \mathbb{C}v\} \subset \pi^* E = \mathbb{P}(E) \times_X E$.

In particular, we have for every $x \in X$ an isomorphism $\mathcal{O}_{\mathbb{P}(E)}(-1)|_{\mathbb{P}(E_x)} \simeq \mathcal{O}_{\mathbb{P}(E_x)}(-1) \simeq \mathcal{O}_{\mathbb{P}^{r-1}}(-1)$.

4.3 Blow-up of a smooth submanifold

If $0 < m < n$ are integer, one can define the blow-up of $\mathbb{C}^m \subset \mathbb{C}^n$ as

$$\mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) := \{([x_{m+1} : \ldots : x_n], z) \in \mathbb{P}^{n-m-1} \times \mathbb{C}^n \mid z_i x_j = z_j x_i \ \forall i,j\}.$$

One can interpret this definition a bit more intrinsically by realizing $\mathbb{P}^{n-m-1}$ as $\mathbb{P}(\mathbb{C}^{n-m})$ the projectivization of the normal bundle of $\mathbb{C}^m$ in $\mathbb{C}^n$ and writing

$$\mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) = \{(\ell, z) \in \mathbb{P}(\mathbb{C}^{m-n}) \times \mathbb{C}^n \mid z \in \langle \mathbb{C}^m, \ell \rangle\}.$$

The second projection $\pi : \mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \to \mathbb{C}^n$ is easily seen to be isomorphic away from $\mathbb{C}^m$, while $\pi^{-1}(z) = \mathbb{P}(\mathbb{C}^{m-n}) \times \{z\}$ if $z \in \mathbb{C}^m$. In particular, $\pi^{-1}(\mathbb{C}^m) = \mathbb{P}(\mathbb{C}^{m-n}) \times \mathbb{C}^m$ is a smooth submanifold of codimension one, that is a smooth hypersurface.

The construction above can be glued in order to define the blow-up of a smooth submanifold $Y \subset X$ of a complex manifold X.

Theorem 4.6. *Let $Y \subset X$ be a smooth submanifold of a complex manifold X. There exists a complex manifold denoted by $\mathrm{Bl}_Y X$, endowed with a proper, surjective holomorphic map $\pi : \mathrm{Bl}_Y X \to X$, that is isomorphic away from Y and such that $E := \pi^{-1}(Y)$ is a smooth hypersurface and the restriction $\pi : E \to Y$ is a holomorphic fiber bundle isomorphic to the projectivized normal bundle $\mathbb{P}(N_{Y|X}) \to Y$.*

Proof. We divide the proof into several step. The starting point is to define $\widetilde{X} := \mathrm{Bl}_Y X$ set theoretically as the disjoint union of $X \setminus Y$ and the total space of $\mathbb{P}(N_Y)$, and $\pi : \widetilde{X} \to X$ is defined as the identity map on $X \setminus Y$ and the projection map on $\mathbb{P}(N_Y)$.

Step 0. *An auxiliary construction.*
Let $U \subset X$ be an open set and let $f, g \in \mathcal{O}_X(U)$ be holomorphic functions vanishing on $Y \cap U$. We define $U_g \subset \pi^{-1}(U) \subset \widetilde{X}$ by

$$U_g := \underbrace{\{z \in U \setminus Y; g(z) \neq 0\}}_{=:U'_g} \cup \underbrace{\{(z,[\xi]) \in \pi^{-1}(Y \cap U); dg_z(\xi) \neq 0\}}_{=:U''_g}.$$

Note that since dg vanishes on T_Y, the set U''_g is well-defined. Moreover, the function $U''_g \ni (z,[\xi]) \mapsto \frac{df_z(\xi)}{dg_z(\xi)}$ is also well-defined. Therefore we have defined a function

$$\widetilde{\frac{f}{g}} : U_g \to \mathbb{C}$$

extending $\frac{f}{g}$ on U_g' to U_g.

Step 1. *The coordinate charts on $\pi^{-1}(U)$.*
Let us fix a coordinate chart $U \subset X$ isomorphic to the unit polydisk such that $Y \cap U = (z_{m+1} = \ldots = z_n = 0)$. Over $Y \cap U$, the normal bundle N_Y is trivial and a frame is given by the class of $(\frac{\partial}{\partial z_{m+1}}, \ldots, \frac{\partial}{\partial z_n})$ modulo T_Y, yielding (coordinate) functions $\xi_{m+1}, \ldots, \xi_n$ on the total space of $N_Y|_{Y\cap U}$. For each $j = m+1, \ldots, n$, we set $U_j = U_{z_j}$ as in the step above. That is,

$$U_j := U_j' \cup U_j'' = \{z \in U \setminus Y \ ; \ z_j \neq 0\} \ \cup \{(z, [\xi]) \in \mathbb{P}(N_Y)|_{Y\cap U} \ ; \ \xi_j \neq 0\}.$$

Note that $\pi^{-1}(U) = U_{m+1} \cup \cdots \cup U_n$. For $x \in U_j$, we set

$$w_k(x) = \begin{cases} z_k(x) & \text{if } 1 \leqslant k \leqslant m \\ z_j(x) & \text{if } k = j \\ \frac{\widetilde{z_k}}{z_j}(x) & \text{if } k \geqslant m+1, k \neq j \end{cases}.$$

This defines a map $\tau_j = (w_1, \ldots, w_n) : U_j \to \mathbb{C}^n$. More precisely, we have for $z \in U \setminus Y$, resp. $(z, [\xi]) \in \mathbb{P}(N_Y)|_{Y\cap U}$:

$$\tau_j(z) = \left(z_1, \ldots, z_m, \frac{z_{m+1}}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, z_j, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j}\right),$$
$$\tau_j(z, [\xi]) = \left(z_1, \ldots, z_m, \frac{\xi_{m+1}}{\xi_j}, \ldots, \frac{\xi_{j-1}}{\xi_j}, 0, \frac{\xi_{j+1}}{\xi_j}, \ldots, \frac{\xi_n}{\xi_j}\right).$$

We infer the following:
- The map τ_j is bijective and $\tau_j(U_j)$ is open. Indeed, the range of τ is

$$\tau_j(U_j) = \{x \in \mathbb{C}^n; |x_j| < 1, |x_j x_k| < 1, \forall k \geqslant m+1\},$$

and its inverse is given by $x \mapsto (x_1, \ldots, x_m, x_{m+1}x_j, \ldots x_{j-1}x_j, x_j, x_{j+1}x_j, \ldots, x_n x_j)$ if $x_j \neq 0$, and $x \mapsto ((x_1, \ldots, x_m), [x_{m+1} : \ldots : x_{j-1} : 1 : x_{j+1} : \ldots : x_n])$ otherwise.
- We have $E \cap U_j = (w_j = 0)$, so that E is a smooth hypersurface of $\widetilde{X}$.
- The map π is holomorphic for the given charts. Indeed, $\pi_j := \tau_j^{-1} \circ \pi$ is given by

$$\pi_j(w_1, \ldots, w_n) = (w_1, \ldots, w_m, w_{m+1}w_j, \ldots, w_{j-1}w_j, w_j, w_{j+1}w_j, \ldots, w_n w_j)$$

which is indeed holomorphic.
- The transition functions $\tau_j \circ \tau_{j'}^{-1} : \tau_{j'}(U_{j'} \cap U_j) \to \tau_j(U_{j'} \cap U_j)$ are holomorphic. This is a straightforward computation.

Step 2. *Compatibility of the charts.*
We are left to checking that the construction above for two open set U, U' with non-empty intersection are compatible. The second set of coordinates $(z_1', \ldots, z_n')$ on X induces new charts $(U_{z_j'}, \tau_j' = (w_1', \ldots, w_n'))$ for $j = m+1, \ldots, n$. We need to prove that for any $j, \ell = m+1, \ldots, n$, the function w_ℓ' is a holomorphic function of $(w_1, \ldots, w_n)$ on $U_j \cap U_{z_\ell'}$.

Given the definition of w'_ℓ, it is enough to show that for any holomorphic functions f, g on U vanishing on $Y \cap U$, the function $\widetilde{\frac{f}{g}} : U_g \to \mathbb{C}$ is a holomorphic function of w when restricted to $U_j \cap U_g$. Now, since f vanishes on $Y \cap U$, one can write $f(z) = \sum_{k=m+1}^n z_k F_k(z)$ for some holomorphic functions F_k on U. In particular, we get on U'_j

$$\begin{aligned} \frac{f(z)}{z_j} &= F_j(z) + \sum_{\substack{m+1 \leqslant k \leqslant n \\ k \neq j}} \frac{z_k}{z_j} F_k(z) \\ &= F_j(\pi_j(w)) + \sum_{\substack{m+1 \leqslant k \leqslant n \\ k \neq j}} w_k F_k(\pi_j(w)) \end{aligned}$$

which admits a holomorphic extension $\widehat{\frac{f}{z_j}} : U_j \to \mathbb{C}$. Now, if $x = (z, [\xi]) \in E \cap U_j$ and $y = \pi(x) \in Y$, we have

$$\widetilde{\frac{f}{z_j}}(x) = \frac{df_z(\xi)}{\xi_j} = \sum_{k=m+1}^{n} \frac{\xi_k}{\xi_j} F_k(y) = \widehat{\frac{f}{z_j}}(x).$$

Therefore, $\widetilde{\frac{f}{z_j}}$ is a holomorphic function in w, non-vanishing on $U_j \cap U_f$. In particular,

$$\widetilde{\frac{f}{g}} = \widetilde{\frac{f}{z_j}} \cdot \left(\widetilde{\frac{g}{z_j}} \right)^{-1}$$

is holomorphic on $U_j \cap U_g$. □

Proposition 4.7. *Let E be the exceptional divisor of the blow-up $\mathrm{Bl}_Y X \to X$. Then the normal bundle of $E = \mathbb{P}(N_Y)$ is isomorphic to $\mathcal{O}_E(-1)$.*

Proof. Set $\widetilde{X} := \mathrm{Bl}_Y X$. The differential of the map π induces a morphism $d\pi : T_{\widetilde{X}} \to \pi^* T_X$. Since $\pi(E) \subset Y$, we have $d\pi(T_E) \subset \pi^* T_Y$ and $d\pi$ induces a map $d\pi : N_E \to \pi^* N_Y$. On U_j, N_E is generated by $\frac{\partial}{\partial w_j}$ while $\pi^* N_Y$ is generated by $\frac{\partial}{\partial z_{m+1}}, \ldots, \frac{\partial}{\partial z_n}$. Moreover, at a point $\tilde{x} = (z, [\xi] = [\sum \xi_k \frac{\partial}{\partial z_k}]) \in U_j \cap E$, we have

$$\begin{aligned} (d\pi)_{\tilde{x}} \left(\frac{\partial}{\partial w_j} \right) &= w_{m+1} \frac{\partial}{\partial z_{m+1}} + \ldots + w_{j-1} \frac{\partial}{\partial z_{j-1}} + \frac{\partial}{\partial z_j} + w_{j+1} \frac{\partial}{\partial z_{j+1}} + \ldots + w_n \frac{\partial}{\partial z_n} \\ &= \frac{\xi_{m+1}}{\xi_j} \frac{\partial}{\partial z_{m+1}} + \ldots + \frac{\xi_{j-1}}{\xi_j} \frac{\partial}{\partial z_{j-1}} + \frac{\partial}{\partial z_j} + \frac{\xi_{j+1}}{\xi_j} \frac{\partial}{\partial z_{j+1}} + \ldots + \frac{\xi_n}{\xi_j} \frac{\partial}{\partial z_n} \\ &= \frac{1}{\xi_j} \sum_{j=m+1}^{n} \xi_k \frac{\partial}{\partial z_k}. \end{aligned}$$

Now recall that $\pi|_E : E \to Y$ is isomorphic to $\mathbb{P}(N_Y) \to Y$ and that by definition of $\mathcal{O}_E(-1) \subset \pi^* N_Y$, $(d\pi)_{\tilde{x}} \left(\frac{\partial}{\partial w_j} \right)$ belongs to $\mathcal{O}_E(-1)_{\tilde{x}}$. This shows that $d\pi : N_E \to \pi^* N_Y$ is an isomorphism onto $\mathcal{O}_E(-1)$. □

Proposition 4.8. *Let X be a compact manifold and let $\pi : \widetilde{X} \to X$ be the blow-up of a smooth submanifold $Y \subset X$ of codimension $r \geqslant 2$, and let E be the exceptional divisor. Given any positive line bundle $L \to X$, the line bundle $\pi^*(L^{\otimes k}) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is positive for $k \gg 1$.*

Proof. Let h_X be an arbitrary hermitian metric on T_X; it induces by restriction to Y and quotient a metric on N_Y and then also a metric h on $\mathcal{O}_{\mathbb{P}(N_Y)}(1)$. For any $y \in Y$, let $E_y := \mathbb{P}(N_{Y,y}) = \pi^{-1}(y)$. The hermitian line bundle $(\mathcal{O}_{\mathbb{P}(N_Y)}(1), h)|_{E_y}$ is isometric to $(\mathcal{O}_{\mathbb{P}^{r-1}}(1), h_{\mathrm{FS}})$ (cf Remark 3.29) hence its Chern curvature form is definite positive.

From Proposition 4.2, $\mathcal{O}_{\widetilde{X}}(-E)|_E \simeq N^*_{E|\widetilde{X}} \simeq \mathcal{O}_E(1)$, hence we can extend h arbitrarily to a metric h_E on $\mathcal{O}_{\widetilde{X}}(-E)$ whose Chern form $\Theta_E := \Theta_h(\mathcal{O}_{\widetilde{X}}(-E))$ is positive along E (i.e. in restriction to E, in the directions of E).

Let h_L be a hermitian metric on L with positive Chern form, and let $F := \pi^*(L^{\otimes k}) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ be endowed with the metric $h_F := \pi^* h_L^{\otimes k} \otimes h_E$. We have for any $t \in T_{\widetilde{X}}$

$$\Theta_{h_F}(F)(t, \bar{t}) = k\Theta_{h_L}(d\pi(t), \overline{d\pi(t)}) + \Theta_E(t, \bar{t}).$$

It is enough to check that the above quantity is positive when $t \in U_{\widetilde{X}}$ is unitary. We know that $\Theta_E(t,t) > 0$ if $t \in T_E \supset \ker(d\pi)$, hence by continuity of Θ_E and compactness of the unitary tangent bundle, there exist $\varepsilon, \delta > 0$ such that $\Theta_E(t, \bar{t}) > \delta$ if $\|d\pi(t)\|_{h_X} < \varepsilon$ (and t is unitary). Finally, the set $K := \{t \in U_{\widetilde{X}}; \|d\pi(t)\|_{h_X} \geqslant \varepsilon\}$ is a compact subset of $U_{\widetilde{X}}$ that does not meet $T_{\widetilde{X}}|_E$. Since $d\pi$ is isomorphic on that set, there exists $\eta > 0$ such that $\Theta_{h_L}(d\pi(t), \overline{d\pi(t)}) > \eta$ for any $t \in K$. Then, one takes $k > \frac{2}{\eta} \sup_K |\Theta_E|$. □

Corollary 4.9. *Let X be a compact manifold and let $\pi : \widetilde{X} \to X$ be a composition of blow-up with smooth centers and let $E = \sum_{i=1}^N E_i$ be the exceptional divisor. Given any positive line bundle $L \to X$, there exist positive numbers $a_1, \ldots, a_N$ such that the line bundle $\pi^*(L^{\otimes k}) \otimes \mathcal{O}_{\widetilde{X}}(-\sum a_i E_i)$ is positive for $k \gg 1$.*

Proof. In order to simplify the notations, we treat the case of two successive blow-ups

$$\widetilde{X} = X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X.$$

Let F_1 (resp. F_2) be the exceptional divisor of π_1 (resp. π_2). The exceptional divisor $E = E_1 + E_2$ of π has too components: E_1 is the strict transform of F_1 by π_2 while $E_2 = F_2$. Note that $\pi_2^* F_1 = E_1 + bE_2$ where $b \geqslant 0$ is positive if and only if F_1 intersects the center of π_2.

By Proposition 4.8, $\pi_1^* L^{k_1} \otimes \mathcal{O}_{X_1}(-F_1)$ is positive for some large $k_1 > 0$. By the same token, $\pi_2^*(\pi_1^* L^{k_1} \otimes \mathcal{O}_{X_1}(-F_1))^{\otimes k_2} \otimes \mathcal{O}_{X_2}(-E_2)$ is positive for some large $k_2 > 0$. But that bundle is nothing but $\pi^*(L^{k_1+k_2}) \otimes \mathcal{O}_{\widetilde{X}}(-k_2 E_1 - (1 + bk_2)E_2)$.

Finally, for $k \geqslant k_1 + k_2$, one sets $k' := k - (k_1 + k_2) \geqslant 0$ and $a_1 := k_2, a_2 := 1 + bk_2$. Then, the decomposition

$$\pi^*(L^{\otimes k}) \otimes \mathcal{O}_{\widetilde{X}}(-\sum a_i E_i) = \pi^*(L^{\otimes k'}) \otimes \left(\pi^*(L^{k_1+k_2}) \otimes \mathcal{O}_{\widetilde{X}}(-\sum a_i E_i)\right),$$

yields the result since the sum of a semipositive form and a positive form is positive. □

4.4 Behavior of positivity notions under blow-ups

In this paragraph, we will state a few basic properties of bimeromorphic maps. We say that a proper map $f : X \to Y$ between complex manifolds is bimeromorphic if there exists a closed analytic proper subset $Z \subsetneq Y$ such that $f : X \setminus f^{-1}(Z) \to Y \setminus Z$ is isomorphic. In particular, X and Y have the same dimension.

The locus $E := \{x \in X;\ \mathrm{rk}(df_x) < n\}$ is either empty or it is an hypersurface in X (i.e. it has codimension one) since it coincides with the zero locus of the holomorphic jacobian $J(f) = \det(\frac{\partial f_i}{\partial z_j})$ in local coordinates (although one could define such a jacobian intrinsically as section of $K_X \otimes f^*K_Y^{-1}$ induced by $df : T_X \to f^*T_Y$).

We will admit a consequence of Zariski's main theorem saying that $E = f^{-1}(f(E))$ and that each fiber of $E \to f(E)$ is connected and positive dimensional. In particular, $\dim(f(E)) \leqslant n-2$.

Proposition 4.10. *Let $f : X \to Y$ be a surjective, bimeromorphic map between compact Kähler manifolds of dimension n. Let $\alpha, \alpha_1, \ldots, \alpha_n \in H^{1,1}(Y, \mathbb{R})$. We have*

1. *$f^*\alpha_1 \cdot \ldots \cdot f^*\alpha_n = \alpha_1 \cdot \ldots \cdot \alpha_n$.*
2. *If α is nef (resp. psef), then so is $f^*\alpha$.*
3. *The class $f^*\alpha$ is Kähler if and only if α is Kähler and f is an isomorphism.*
4. *If α is big and f is a composition of blow-ups with smooth centers, then $f^*\alpha$ is big.*

Proof. The first item is easy: choose smooth representatives $\theta_i \in \alpha_i$. Since $f^{-1}(Z)$ has measure zero, the change of variables formula implies

$$\begin{aligned} f^*\alpha_1 \cdot \ldots \cdot f^*\alpha_n &= \int_X f^*\theta_1 \wedge \ldots \wedge f^*\theta_n \\ &= \int_{X \setminus f^{-1}(Z)} f^*\theta_1 \wedge \ldots \wedge f^*\theta_n \\ &= \int_{Y \setminus Z} \theta_1 \wedge \ldots \wedge \theta_n \\ &= \int_Y \theta_1 \wedge \ldots \wedge \theta_n \\ &= \alpha_1 \cdot \ldots \cdot \alpha_n. \end{aligned}$$

For 2, we fix a positive definite form ω_X on X (resp. ω_Y on Y). Up to rescaling ω_Y, one can assume that $f^*\omega_Y \leqslant \omega_X$. Now, if α is nef, there exists for each $\varepsilon > 0$ a representative $\theta_\varepsilon \in \alpha$ satisfies $\theta_\varepsilon \geqslant -\varepsilon\omega_Y$, then $f^*\alpha \ni f^*\theta_\varepsilon \geqslant -\varepsilon f^*\omega_Y \geqslant -\varepsilon\omega_X$ and $f^*\alpha$ is nef. If α is psef, it contains a positive current T, hence $f^*\alpha$ contains the positive current f^*T, cf Proposition 3.17.

As for 3, assume that f is not isomorphic. In particular, the exceptional divisor E is non-empty and $f(E)$ has dimension at most $n-2$. In particular, $(f^*\alpha^{n-1})|_E = f^*(\alpha^{n-1})|_{f(E)}$ is zero. Now, by Corollary 3.31, we have $f^*\alpha^{n-1} \cdot c_1(E) = (f^*\alpha^{n-1})|_E = 0$. However, if $f^*\alpha$ were to contain a Kähler form $\theta > 0$, one would have $f^*\alpha^{n-1} \cdot c_1(E) = \int_E \theta|_E^{n-1} > 0$.

Finally, let us prove 4. It is sufficient to prove the claim for α Kähler. By an immediate generalization of Corollary 4.9 to case of $(1,1)$ classes, we see that $f^*\alpha - \sum a_i c_1(E_i)$ contains a Kähler form θ for some numbers $a_i \geqslant 0$. In particular, $\theta + \sum a_i [E_i]$ is a positive Kähler current in $f^*\alpha$. □

4.5 The example of $\mathrm{Bl}_0\mathbb{P}^2$.

In this paragraph, we work on the fixed manifold $X = \mathrm{Bl}_0\mathbb{P}^2$, the blow-up of the projective plane at one point. The choice of the point is irrelevant (since $\mathrm{PGL}(3,\mathbb{C})$ acts transitively on $\mathbb{P}^2$). Let $f : X \to \mathbb{P}^2$ be the blow-down map.

We set $\alpha = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$, this is a Kähler class containing the Fubini-Study Kähler metric ω_{FS}. Let E be the exceptional divisor.

Lemma 4.11. *We have the following:*

1. *The class $f^*\alpha$ is nef and big, but not Kähler.*
2. *For any $b > 0$, the class $f^*\alpha + bE$ is big but not nef.*

Proof. The first item follows from Proposition 4.10. We move on to proving the second assertion now. As a manifold $E \simeq \mathbb{P}^1$, and its normal bundle $N_{E|X} \simeq \mathcal{O}_X(E)|_E \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$. By Proposition 3.31, we get

$$E \cdot E = \int_E c_1(E)|_E = \int_E c_1(\mathcal{O}_X(E)|_E) = \int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(-1)) = \int_{\mathbb{P}^1} (-\omega_{\mathrm{FS}}) = -1. \tag{4.13}$$

Since $f^*\alpha$ is big, so is $f^*\alpha + bE$ for any $b \geqslant 0$. Since E is mapped to a point, we have $f^*\alpha|_E = f^*(\alpha|_E) = 0$ hence

$$f^*\alpha \cdot E = \int_E f^*\alpha|_E = 0. \tag{4.14}$$

In particular, we get

$$(f^*\alpha + bE) \cdot E = -bE \cdot E = -b$$

and $f^*\alpha + bE$ is never nef when $b > 0$, cf Lemma 3.26. We could also have used the computation, using Proposition 4.10 and the identities (4.13)-(4.14) above:

$$(f^*\alpha + bE)^2 = f^*\alpha^2 + 2bf^*\alpha \cdot E + b^2E^2 = \alpha^2 - b^2 = 1 - b^2$$

which is negative as soon as $b > 1$. □

5 The big cone

5.1 Currents with analytic singularities

Let X be a complex manifold.

Definition 5.1. A closed, quasi-positive $(1,1)$-current T on X is said to have analytic singularities along a subvariety $V = V(\mathcal{I})$ if locally, when $\mathcal{I} = (f_1, \ldots, f_r)$ one can write

$$T = (\text{smooth form}) + \alpha dd^c \log \left(\sum_{k=1}^{r} |f_k|^2 \right)$$

for some $\alpha > 0$.
We denote by $Z = Z(T)$ the singular locus of T, i.e. (the support of) V.

Example 5.2. A current of integration along a smooth hypersurface $E \subset X$ has analytic singularities by Lelong-Poincaré formula $dd^c \log |f|^2 = [f = 0]$. However, as soon as the subvariety V has codimension greater than or equal to 2, a current with analytic singularities along V is not supported on V anymore.

Example 5.3 (Global examples)**.** On $\mathbb{P}^n$, it is easy to construct such currents. Indeed, let $\mathcal{I} = (f_1, \ldots, f_r)$ be an homogeneous ideal of degree d. We can see f_i as a section of $L := \mathcal{O}_{\mathbb{P}^n}(d)$, and then define $T := \omega_{\text{FS}} + \alpha dd^c \log \left(\sum_{i=1}^r |f_i|^2_{h_{\text{FS}}} \right)$. This is a positive current if and only if $\alpha d < 1$.
The construction generalizes to projective manifolds in order to construct positive currents with analytic singularities (in positive rational classes).

Example 5.4. Let X be a compact manifold and let $Z \subset X$ be a Zariski closed subset. We can cover X by chart U_α where $Z \cap U_\alpha = (f_1^\alpha = \ldots = f_N^\alpha = 0)$ (one can pick the same N for all charts up to adding redundancy). Let χ_α be a partition of unity subordinate to (U_α). Then one check easily that $\psi := \log \left(\sum_\alpha \chi_\alpha \sum_{i=1}^N |f_i^\alpha|^2 \right)$ is a quasi-psh function such that $\{\psi = -\infty\} = Z$.

Theorem 5.5 (Demailly's regularization theorem)**.** *Let X be a compact complex manifold, let $T = \theta + dd^c \varphi$ be a closed almost positive $(1,1)$-current, i.e. θ is a closed smooth $(1,1)$-form and φ is quasi-psh. Let γ be a smooth $(1,1)$-form such that $T \geqslant \gamma$. Let ω be a fixed positive $(1,1)$-form.*

There exists a decreasing sequence of quasi-psh functions (φ_k) such that

1. *$T_k := \theta + dd^c \varphi_k$ has analytic singularities*
2. *$\varphi_k \downarrow \varphi$ as $k \to +\infty$. In particular, $T_k \to T$ weakly.*
3. *$T_k \geqslant \gamma - \varepsilon_k \omega$ for some sequence $\varepsilon_k \downarrow 0$.*

The proof is quite involved, but it is easier to explain what happens locally. If φ is a psh function on a open set $U \subset \mathbb{C}^n$, then one can consider for any integer $m \geqslant 1$ the Hilbert space $H_{m\varphi} = \{f \in \mathcal{O}(U) \cap L^2(U, e^{-m\varphi})\} = \{f \in \mathcal{O}(U); \int_U |f|^2 e^{-m\varphi} <$

$+\infty\}$. Choose a Hilbert basis $(f_k^{(m)})$ of $H_{m\varphi}$ and set $\varphi_m := \frac{1}{m}\log\left(\sum_{k=1}^{+\infty}|f_k^{(m)}|^2\right)$. One can check that φ_m is well-defined, psh with analytic singularities, and almost decreasing. The difficult task is then to show that φ_m converges to φ as $m \to +\infty$.

Corollary 5.6. *Let X be a compact Kähler manifold, and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class. There exists a Kähler current $T \in \alpha$ with analytic singularities.*

Proof. Let $T = \theta + dd^c\varphi$ be a Kähler current in α; that is, $T \geqslant \omega$ for some Kähler form ω. The currents T_k produced by Demailly's regularization satisfy $T_k \geqslant (1-\varepsilon_k)\omega$ hence they are Kähler for $k \gg 1$. □

Definition 5.7. Let X be a compact Kähler manifold, and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class. Let $\mathcal{S}$ be the set of Kähler currents in α with analytic singularities. The non-Kähler locus $E_{\mathrm{nK}}(\alpha)$ is defined as

$$E_{\mathrm{nK}}(\alpha) = \bigcap_{T\in\mathcal{S}} Z(T).$$

Proposition 5.8 (Boucksom). *The non-Kähler locus $E_{\mathrm{nK}}(\alpha)$ of a big class is an analytic subvariety of X, and there exists $T_0 \in \mathcal{S}$ such that $E_{\mathrm{nK}}(\alpha) = Z(T_0)$. Moreover, $E_{\mathrm{nK}}(\alpha) = \emptyset$ if and only if α is Kähler.*

Proof. Let $T_i = \theta + dd^c\varphi_i \in \mathcal{S}$ for $i = 1, 2$. Defined $\varphi_3 = \max(\varphi_1, \varphi_2)$. Then $T_3 := \theta + dd^c\varphi_3$ is again a Kähler current and $(\varphi_3 = -\infty) \subset Z(T_1) \cap Z(T_2)$. By Demailly regularization, one can find a Kähler current $T = \theta + dd^c\varphi$ with analytic singularities such that $\varphi \geqslant \varphi_3$. In particular, $Z(T) \subset (\varphi_3 = -\infty) \subset Z(T_1) \cap Z(T_2)$.

Now, if the claim fails, given an initial $T_1 \in \mathcal{S}$, we have $E \subsetneq Z(T_1)$. Therefore, there exists $T_1' \in \mathcal{S}$ such that $Z(T_1) \subsetneq Z(T_1')$. By the above, we can find T_2 such that $Z(T_2) \subset Z(T_1) \cap Z(T_1') \subsetneq Z(T_1)$. Iterating the construction, we find a decreasing sequence of analytic subsets, which has to be stationary by the Noetherian property. The "limit" current T_0 satisfies $E_{\mathrm{nK}}(\alpha) = Z(T_0)$.

Finally, if $E_{\mathrm{nK}}(\alpha) = \emptyset$, then there exists a Kähler current T_0 with analytic singularities such that $Z(T_0) = \emptyset$; that is, T_0 is a Kähler form. □

Theorem 5.9 (Hironaka's theorem). *Let X be a complex manifold and let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent ideal sheaf. There exists a sequence of blow-ups with smooth centers $\pi : \widetilde{X} \to X$ such that $\pi^{-1}\mathcal{I} = \mathcal{O}_X(-D)$ for some divisor D in $\widetilde{X}$.*

Recall that if $\mathcal{I} = (f_1, \ldots, f_r)$ locally, then $\pi^{-1}\mathcal{I}$ is defined by $(f_1 \circ \pi, \ldots, f_r \circ \pi)$. We call $\pi : \widetilde{X} \to X$ a resolution of singularities of $(X, \mathcal{I})$.

Let T be a current with analytic singularities along $V(\mathcal{I})$ and if $\pi : \widetilde{X} \to X$ is a resolution of singularities of $(X, \mathcal{I})$. Let $x \in D$, and let U be a neighborhood of x in $\widetilde{X}$ such that $D \cap U = (g = 0)$. We can find functions $g_1, \ldots, g_r \in \mathcal{O}_{\widetilde{X}}(U)$ such that $\pi^* f_i = g g_i$ with $(g_1, \ldots, g_r) = \mathcal{O}_{\widetilde{X}}$, i.e. $(g_1 = \ldots = g_r = 0) = \emptyset$. On U, we thus find

$$\pi^*T|_U = (\text{smooth form}) + \alpha dd^c \log|g|^2$$

hence we get globally on $\widetilde{X}$

$$\pi^*T = \beta + \alpha[D],$$

where β is a smooth, closed $(1,1)$-form on $\widetilde{X}$. Moreover, we have $T \geqslant 0$ if and only if $\beta \geqslant 0$.

5.2 Currents with minimal singularities

Let X be a compact Kähler manifold, and let θ be a smooth, closed $(1,1)$-form. We set $\alpha := [\theta] \in H^{1,1}(X,\mathbb{R})$ and define

$$V_\theta := \sup\{\psi \leqslant 0; \psi \in \mathrm{PSH}(X,\theta)\}.$$

Lemma 5.10. *The function V_θ is not identically $-\infty$ if and only if α is pseudoeffective. In that case, $V_\theta \in \mathrm{PSH}(X,\theta)$.*

Proof. The set in the supremum is non-empty if and only if α admits a positive current, hence the claim. As for the second fact, we can assume that V_θ is obtained as an increasing limit of a sequence of θ-psh functions. It is then quasi-psh if and only if it is usc. But if V_θ is not usc, its upper continuous regularization V_θ^* satisfies $V_\theta \leqslant V_\theta^* \leqslant 0$ and $V_\theta^* \in \mathrm{PSH}(X,\theta)$, hence $V_\theta^* \leqslant V_\theta$ and we get equality. □

If $\varphi \in \mathrm{PSH}(X,\theta)$, then $\varphi - \sup_X \varphi \leqslant V_\theta$, hence $\varphi \leqslant V_\theta + \sup_X \varphi$ is "more singular" than V_θ.

Definition 5.11. A function $\varphi \in \mathrm{PSH}(X,\theta)$ is said to have minimal singularities if $\varphi \geqslant V_\theta + O(1)$. In that case, $\varphi = V_\theta + O(1)$. A positive current $T = \theta + dd^c\varphi \in \alpha$ has minimal singularities if φ has minimal singularities.

Definition 5.12. Given two closed positive currents $T = \theta + dd^c\varphi$, $T' = \theta + dd^c\varphi'$, we say that T is less singular than T' if $\varphi' \leqslant \varphi + O(1)$. In particular, a positive current with minimal singularities is less singular than any other positive current in the the same cohomology class.

Example 5.13. Let $f : X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at one point, let E be the exceptional divisor, let s be the canonical section of $\mathcal{O}_X(E)$ and let h be a hermitian metric on $\mathcal{O}_X(E)$ with curvature form θ, normalized so that $\sup_X \log|s|_h^2 = 0$. Then $V_\theta = \log|s|_h^2$.

Example 5.14. Let $f : X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at one point, let E be the exceptional divisor and let $\alpha = f^*[\omega_{\mathrm{FS}}]$. For any $b \geqslant 0$, the current $f^*\omega + b[E]$ has minimal singularities. Indeed, let θ_E be the curvature of a hermitian metric h on $\mathcal{O}_X(E)$ and let $T = f^*\omega + b\theta_E + dd^c\varphi$ be a positive current. Then $\varphi - b\log|s|_h^2$ is $f^*\omega_{\mathrm{FS}}$-psh on $X \setminus E \simeq \mathbb{P}^2 \setminus \{pt\}$, hence it is bounded above near the point. In particular, $\varphi \leqslant b\log|s|_h^2 + O(1)$ on $X \setminus E$, hence on X globally since quasi-psh function are determined by their data almost everywhere.

Example 5.15. Let α, α' be big classes and let T, T' be positive currents with minimal singularities in α (resp. α'). Then $T + T'$ needs not have minimal singularities. An easy example is with $X = \mathrm{Bl}_0\mathbb{P}^2$, $\alpha = f^*[\omega_{\mathrm{FS}}] - \varepsilon E$, $\alpha' = \varepsilon E$. For small $\varepsilon > 0$, T will have bounded potentials while T' will be singular along $[E]$; hence so will $T + T'$ although $\alpha + \alpha'$ is semipositive.

Remark 5.16. The type of singularity of V_θ (i.e. the set $\{V_\theta + f, f \in L^\infty(X)\}$) only depends on $[\theta]$. If $[\theta]$ is semipositive, then V_θ is bounded. It may not be the case if $[\theta]$ is only assumed to be nef.

The following proposition summarizes a few properties of currents with minimal singularities.

Proposition 5.17. *Let X be a compact Kähler manifold, let α be a psef class and let $\theta \in \alpha$ a smooth representative.*

1. *If α is big, then V_θ is locally bounded on the Zariski open set $X \setminus E_{\mathrm{nK}}(\alpha)$.*
2. *If α contains a Kähler current with minimal singularities, then α is Kähler.*
3. *If $f : Y \to X$ is a proper surjective bimeromorphic map, then $V_{f^*\theta} = f^* V_\theta$.*

Proof. The first point is obvious since any current T with analytic singularities along $E_{\mathrm{nK}}(\alpha)$ has locally bounded potentials away from that set.

The second item is due to Boucksom and goes as follows. Let $T = \theta + dd^c\varphi$ be such a current. For ε small enough, $\theta + dd^c(1-\varepsilon)\varphi = (1-\varepsilon)T + \varepsilon\theta$ is still a Kähler current hence it is positive and $(1-\varepsilon)\varphi \geqslant \varphi + O(1)$. This implies that φ is bounded. Another regularization theorem of Demailly then ensures that one can regularize φ into a smooth strictly θ-psh function.

As for the last point, we have seen in Proposition 3.17 that f induces an isomorphism $f^* : \mathrm{PSH}(X,\theta) \to \mathrm{PSH}(Y, f^*\theta)$. The conclusion follows. □

5.3 Non-pluripolar Monge-Ampère product

Recall that Bedford and Taylor defined a Monge-Ampère operator for *bounded* psh functions u by iterating $(dd^c u)^2 = dd^c(u(dd^c u)), \ldots, (dd^c u)^n = dd^c(u(dd^c u)^{n-1})$. The resulting current $(dd^c u)^n$ is a positive measure which satisfies the following crucial property: if $u_k \downarrow u$ is a sequence of psh functions decreasing to u, then $(dd^c u_k)^n \to (dd^c u)^n$ weakly. Moreover, $(dd^c u)^n$ puts no mass on pluripolar sets (in particular, on proper analytic sets).

Back to the global setting, let θ be a closed $(1,1)$-form. If $\varphi \in \mathrm{PSH}(X,\theta) \cap L^\infty(X)$, one can define the Monge-Ampère measure $(\theta + dd^c\varphi)^n$ using Bedford-Taylor theory. More precisely, one can defined $(\theta + dd^c\varphi)^n$ inductively by setting

$$(\theta + dd^c\varphi)^k = \theta \wedge (\theta + dd^c\varphi)^{k-1} + dd^c(\varphi(\theta + dd^c\varphi)^{k-1}).$$

By Stokes theorem, it is easy to see that

$$\int_X (\theta + dd^c\varphi)^n = \int_X \theta^n.$$

For instance, if $n = 2$, one has $(\theta + dd^c\varphi)^2 = \theta^2 + 2\theta \wedge dd^c\varphi + dd^c(\varphi dd^c\varphi)$. The last two summands are d-exact $(2,2)$-currents, hence their integral is zero.

Let us consider a more general situation where $\varphi \in \mathrm{PSH}(X,\theta)$ is such that $\varphi \in L^\infty_{\mathrm{loc}}(U)$ for some Zariski-open subset $U \subset X$. We set $Z := X \setminus U$. A typical case is when φ has analytic singularities, or when φ has minimal singularities and $[\theta]$ is big.

Proposition 5.18. *There exists $C = C([\theta])$ such that for any $\varphi \in \mathrm{PSH}(X,\theta) \cap \in L^\infty_{\mathrm{loc}}(U)$, one has*

$$\int_U (\theta + dd^c\varphi)^n \leqslant C.$$

In particular, the integral is finite.

Proof. Let ω be a Kähler form on X. There exists $C > 0$ such that $\alpha + C[\omega]$ is Kähler. Since $\theta + dd^c\varphi \leqslant \theta + C\omega + dd^c\varphi$, one can assume that α is Kähler. In order to simplify the notation, one will assume that θ itself is Kähler, although we don't actually need it.

For $k \geqslant 0$, set $\varphi_k := \max(\varphi, -k)$. This is a decreasing sequence of θ-psh functions. In particular, we have for any relatively compact set $V \Subset U$:

$$\int_V (\theta + dd^c\varphi)^n \leqslant \liminf \int_V (\theta + dd^c\varphi_k)^n \leqslant \liminf \int_X (\theta + dd^c\varphi_k)^n = \int_X \theta^n.$$

Since $V \subset U$ is arbitrary large, we get the result. □

Definition 5.19. The (non-pluripolar) Monge-Ampère measure $\langle(\theta + dd^c\varphi)^n\rangle$ of a function $\varphi \in \mathrm{PSH}(X,\theta)$ which is locally bounded on some Zariski-open subset $U \subset X$ is the trivial extension of the positive measure $(\theta + dd^c\varphi)^n$ on U. It has finite mass by Proposition 5.18.

Remark 5.20. The construction shows that more generally, given $T_1, \ldots, T_p$ closed positive $(1,1)$-currents on X which are locally bounded on some Zariski-open subset $U \subset X$, then the trivial extension of $T_1 \wedge \ldots \wedge T_p$ yields a positive (p,p)-current $\langle T_1 \wedge \ldots \wedge T_p\rangle$ with finite mass on X. Moreover, it is a closed current by Skoda-El Mir theorem. In particular, it induces a cohomology class $[\langle T_1 \wedge \ldots \wedge T_p\rangle] \in H^{p,p}(X,\mathbb{R})$.

Example 5.21. Let E be an hypersurface on X, and let θ be the curvature of some hermitian metric h on $\mathcal{O}_X(E)$. The canonical section s of E induces a θ-psh function $\varphi = \log|s|_h^2$ such that $\theta + dd^c\varphi = [E]$. The function φ is smooth outside E, and its Monge-Ampère vanishes there. Hence $\langle(\theta + dd^c\varphi)^n\rangle \equiv 0$.

We can construct slightly more involved examples where mass is only partially lost. For instance, let $E = (\sum_{i=0}^n z_i^2 = 0) \subset \mathbb{P}^n$ be a smooth quadric, and let $s = \sum z_i^2$ seen as section of $\mathcal{O}_{\mathbb{P}^n}(2)$. Define $\varphi = \frac{1}{2}\log|s|^2_{h_{\mathrm{FS}}^{\otimes 2}}$; it satisfies $2\omega_{\mathrm{FS}} + dd^c\varphi = \omega_{\mathrm{FS}} + \frac{1}{2}[E]$, hence $\langle(2\omega_{\mathrm{FS}} + dd^c\varphi)^n\rangle = \omega_{\mathrm{FS}}^n$ which has mass

$$\int_{\mathbb{P}^n} \langle(2\omega_{\mathrm{FS}} + dd^c\varphi)^n\rangle = \int_X \omega_{\mathrm{FS}} = \frac{1}{2^n}\int_X (2\omega_{\mathrm{FS}})^n.$$

Theorem 5.22 (Boucksom-Eyssidieux-Guedj-Zeriahi)**.** *Let X be a compact Kähler manifold and let T, T' be two positive currents in the same cohomology class, locally bounded on a Zariski open subset, and such that T' is more singular than T. Then*

$$\int_X \langle T'^n\rangle \leqslant \int_X \langle T^n\rangle.$$

Proof. One proves the following statement instead. For any $T_i, T_i{}'$ positive currents as in the assumptions, then one has the following inequality in cohomology

$$[\langle T_1' \wedge \ldots \wedge T_p'\rangle] \leqslant [\langle T_1 \wedge \ldots \wedge T_p\rangle],$$

i.e. the difference contains a positive (p,p)-current. By duality, this means that for any d-closed, positive smooth $(n-p, n-p)$-form τ, one has

$$\int_X \langle T_1' \wedge \ldots \wedge T_p' \rangle \wedge \tau \leqslant \int_X \langle T_1 \wedge \ldots \wedge T_p \rangle \wedge \tau.$$

By induction, one can assume that for $i \geqslant 2$, one has $T_i = T_i'$. One sets $T := T_1 = \theta + dd^c\varphi, T' := T_1' = \theta + dd^c\varphi'$, one picks an arbitrary smooth semipositive $(n-p, n-p)$-form τ, and one sets $\Theta := \langle T_2 \wedge \ldots \wedge T_r \rangle \wedge \tau$, which is a closed, positive $(n-1, n-1)$-current. Everything now comes down to showing that

$$\int_U dd^c\varphi' \wedge \Theta \leqslant \int_U dd^c\varphi \wedge \Theta,$$

where U is a Zariski open subset on which φ, φ' are locally bounded and satisfy $\varphi' \leqslant \varphi$.

Let ψ be a quasi-psh function such that $Z := X \setminus U = \{\psi = -\infty\}$. By the weak convergence $dd^c(\varphi' + \varepsilon\psi) \wedge \Theta \to dd^c\varphi' \wedge \Theta$ and the fact that mass does not increase at the limit, it is enough to show the statement for $\varphi' + \varepsilon\psi$. That is, one can assume that $\varphi' - \varphi \to -\infty$ near Z.

For k fixed, set $\psi_k = \max(\varphi', \varphi - k)$ so that $\psi_k \equiv \varphi - k$ in a neighborhood of Z (depending on k). The current $S := d^c(\psi_k - \varphi) \wedge \Theta$ has compact support in U and can be extended to X trivially. By Stokes theorem, $dd^c(\psi_k - \varphi) \wedge \Theta = dS$ has integral zero on X, hence on U. That is,

$$\int_U dd^c\psi_k \wedge \Theta = \int_U dd^c\varphi \wedge \Theta.$$

Now, ψ_k converges to φ', hence $dd^c\psi_k \wedge \Theta \to dd^c\varphi' \wedge \Theta$ weakly on U. The sought inequality follows. □

As a consequence, if T, T' have minimal singularities in a given big class, then they are locally bounded on a Zariski open subset and $\int_X \langle T^n \rangle = \int_X \langle T'^n \rangle$.

Definition 5.23 (Volume of a big class)**.** Let X be a compact Kähler manifold of dimension n and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class. The volume of α, denoted by $\mathrm{vol}(\alpha)$, is the total non-pluripolar Monge-Ampère mass

$$\mathrm{vol}(\alpha) := \int_X \langle T_{\min}^n \rangle$$

of any positive current $T_{\min} \in \alpha$ with minimal singularities.

More generally, we have showed that if $T_i \in \alpha_i$ have minimal singularities in the big class α_i, then the product

$$\langle \alpha_1 \ldots \alpha_p \rangle := [\langle T_1 \wedge \ldots \wedge T_p \rangle] \in H^{p,p}(X, \mathbb{R})$$

is well-defined, i.e. it depends only on the cohomology classes $\alpha_1, \ldots, \alpha_p$. If $p = n$, then $\langle \alpha_1 \ldots \alpha_n \rangle = \int_X \langle T_1 \wedge \ldots \wedge T_n \rangle \in \mathbb{R}_+$ is a number.

However, the product $\langle \alpha_1 \ldots \alpha_n \rangle$ is not multi-linear, since the sum of two currents with minimal singularities needs not have minimal singularities. Yet, we have the following monotonicity property

Proposition 5.24. *Let α, α' be big classes such that $\alpha' \geqslant \alpha$, in the sense that $\alpha' - \alpha \in \mathcal{E}$. Then,* $\mathrm{vol}(\alpha) \leqslant \mathrm{vol}(\alpha')$.

Proof. We reformulate the statement as follows. Let α be a big class and let α' be a psef class. Then $\mathrm{vol}(\alpha) \leqslant \mathrm{vol}(\alpha + \alpha')$.

We let $\theta \in \alpha$ (resp. $\theta' \in \alpha'$) be a smooth representative, and we let $T = \theta + dd^c\varphi \in \alpha$ (resp. $T' = \theta' + dd^c\varphi' \in \alpha'$) be a positive current with minimal singularities (resp. a positive current).

We know that $V_\theta, V_{\theta+\theta'}$ are bounded on a Zariski open set $U \subset X$. By definition

$$\mathrm{vol}(\alpha) = \int_U (\theta + dd^c\varphi)^n = \lim_{k\to+\infty} \int_{U_k} (\theta + dd^c\varphi)^n \tag{5.15}$$

for any increasing sequence of measurable sets $U_k \uparrow U$.

We choose $U_k = U \cap (\psi_k > V_{\theta+\theta'} - k)$ where $\psi_k = \max(\varphi + \varphi', V_{\theta+\theta'} - k)$. Note that U_k may not be an open set. However, ψ_k and $\varphi + \varphi'$ are locally bounded on U_k and Bedford and Taylor proved that

$$1_{U_k}(\theta + \theta' + dd^c\psi_k)^n = 1_{U_k}(\theta + \theta' + dd^c(\varphi + \varphi'))^n. \tag{5.16}$$

The inequality of currents $\theta + dd^c\varphi \leqslant \theta + \theta' + dd^c(\varphi + \varphi')$ combined with (5.16) allows us to say that the respective Bedford-Taylor products satisfy

$$1_{U_k}(\theta + dd^c\varphi)^n \leqslant 1_{U_k}(\theta + \theta' + dd^c\psi_k)^n. \tag{5.17}$$

Now, $\psi_k \in \mathrm{PSH}(X, \theta + \theta')$ has minimal singularities, hence $\int_U (\theta + \theta' + dd^c\psi_k)^n = \mathrm{vol}(\alpha + \alpha')$. Combining (5.15) and (5.17), we get

$$\begin{aligned}
\mathrm{vol}(\alpha) &= \lim_{k\to+\infty} \int_{U_k} (\theta + dd^c\varphi)^n \\
&\leqslant \lim_{k\to+\infty} \int_{U_k} (\theta + \theta' + dd^c\psi_k)^n \\
&\leqslant \lim_{k\to+\infty} \int_{U} (\theta + \theta' + dd^c\psi_k)^n \\
&= \mathrm{vol}(\alpha + \alpha').
\end{aligned}$$

□

Corollary 5.25. *The volume functional is continuous on the big cone $\mathring{\mathcal{E}}$ and it can be extended continuously to the boundary of the psef cone. Moreover, we have $\mathrm{vol}(\alpha) = \alpha^n$ whenever α is nef.*

Proof. The volume is homogeneous and non-decreasing with respect to the partial order induced by positive currents thanks to the lemma above. Fix a big class α and a Kähler form ω. One can assume that $\alpha \geqslant [\omega]$.

If $\alpha_k \to \alpha$, one can find $\varepsilon_k \to 0$ such that $\alpha - \varepsilon_k[\omega] \leqslant \alpha_k \leqslant \alpha + \varepsilon_k[\omega]$, hence $(1 - \varepsilon_k)\alpha \leqslant \alpha_k \leqslant (1 + \varepsilon_k)\alpha$ and the result follows.

Similarly, if $\alpha \in \partial\mathcal{E}$, then $\mathrm{vol}(\alpha + \varepsilon[\omega])$ is non-increasing when $\varepsilon \to 0$, and the limit is independent of ω.

As for the last statement, it follows from the continuity of the volume and the intersection numbers that it is enough to show it for α Kähler. But then, any Kähler form $\omega \in \alpha$ has minimal singularities, hence $\mathrm{vol}(\alpha) = \int_X \omega^n = \alpha^n$. □

Remark 5.26. More generally, if $\alpha_1, \dots, \alpha_p$ are nef, then $\langle \alpha_1 \dots \alpha_p \rangle = (\alpha_1 \dots \alpha_p)$.

Lemma 5.27. *Let α be a Kähler class and let β be a big class. Then, one has*

$$(\alpha \cdot \langle \beta^{n-1} \rangle) = \langle \alpha \cdot \beta^{n-1} \rangle.$$

Proof. Let $\omega \in \alpha$ be a Kähler class, and let $T \in \beta$ be a positive current with minimal singularities. By definition, the RHS is $\int_{X \setminus Z} \omega \wedge T^{n-1}$ where Z is the complement of the locus where T has locally bounded potentials. Said otherwise, the RHS is equal to $\int_X \omega \wedge \langle T^{n-1} \rangle$ where $\langle T^{n-1} \rangle$ is the extension by zero of T^{n-1} across Z. By definition, $\langle \beta^{n-1} \rangle = [\langle T^{n-1} \rangle] \in H^{n-1,n-1}(X, \mathbb{R})$ so that the RHS coincides with $[\omega] \cdot [\langle T^{n-1} \rangle] = (\alpha \cdot \langle \beta^{n-1} \rangle)$. $\square$

Lemma 5.28. *Let $f : Y \to X$ be a surjective bimeromorphic map between compact Kähler manifolds and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class. Then* $\mathrm{vol}(f^*\alpha) = \mathrm{vol}(\alpha)$.

Proof. Let $T \in \alpha$ be a positive current with minimal singularities. From Proposition 5.17, $f^*T \in f^*\alpha$ has minimal singularities.

There exists a Zariski open set U such that T has locally bounded potentials on U and f induces an isomorphism $f : f^{-1}(U) \to U$. In particular we have $\mathrm{vol}(f^*\alpha) = \int_{f^{-1}(U)} (f^*T)^n = \int_U T^n = \mathrm{vol}(\alpha)$. $\square$

Note that we are hiding in the proof the fact that $f^*\alpha$ is big, which we have proved only when f is a composition of blow ups with smooth centers.

We will admit the following theorem and only give a proof when the classes at stake are nef.

Theorem 5.29 (Hovanskii-Teissier inequalities)**.** *Let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be psef classes, and let $1 \leqslant p \leqslant n$ be an integer. Then, one has*

$$\langle \alpha^p \cdot \beta^{n-p} \rangle \geqslant \langle \alpha^n \rangle^{\frac{p}{n}} \cdot \langle \beta^n \rangle^{\frac{n-p}{n}}.$$

Proof. We assume that α, β are nef. By continuity, one can assume that they are Kähler. Let $\omega \in \alpha$ and $\omega' \in \beta$ be representatives. By homogeneity, one can assume that $\beta^n = 1$. We claim that

$$\omega^p \wedge \omega'^{n-p} \geqslant \left(\frac{\omega^n}{\omega'^n} \right)^{\frac{p}{n}} \omega'^n. \tag{5.18}$$

Indeed, this can be check at a point x where $\omega = \sum_{k=1}^n \lambda_k i dz_k \wedge d\bar{z}_k$ and $\omega' = \sum_{k=1}^n i dz_k \wedge d\bar{z}_k$. The LHS becomes $\left(\sum_{i_1 < \ldots < i_p} \lambda_{i_1} \cdots \lambda_{i_p} \right) \omega'^n$ while the RHS is simply $(\lambda_1 \cdots \lambda_n)^{\frac{p}{n}} \omega'^n$. Reorder the indices so that $\lambda_1 \geqslant \cdot \geqslant \lambda_n$. We have

$$\begin{aligned}
\left(\sum_{i_1 < \ldots < i_p} \lambda_{i_1} \cdots \lambda_{i_p} \right)^n &\geqslant (\lambda_1 \cdots \lambda_p)^n \\
&\geqslant (\lambda_1 \cdots \lambda_p)^p \cdot \lambda_p^{p(n-p)} \\
&\geqslant (\lambda_1 \cdots \lambda_p)^p \cdot (\lambda_{p+1} \cdots \lambda_n)^p \\
&= (\lambda_1 \cdots \lambda_n)^p.
\end{aligned}$$

Finally, set $V = \alpha^n$; by Yau's theorem, one can find a (new) representative $\omega \in \alpha$ such that $\omega^n = V\omega'^n$. Apply (5.18) to ω, ω' and integrate over X to get

$$\begin{aligned}(\alpha^p \cdot \beta^{n-p}) &= \int_X \omega^p \wedge \omega'^{n-p} \\ &\geqslant \int_X V^{\frac{p}{n}} \omega'^n = (\alpha^n)^{\frac{p}{n}}.\end{aligned}$$

□

We end this section with the follow characterization of big classes due to Boucksom, and relying on the method of concentration of Monge-Ampère mass introduced by Demailly-Păun:

Theorem 5.30 (Boucksom)**.** *Let X be a compact Kähler manifold and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudoeffective class. Then α is big if and only if* $\mathrm{vol}(\alpha) > 0$.

In other words, the volume functional vol on the psef cone $\mathcal{E}$ coincides with the extension by zero of $\mathrm{vol}|_{\mathring{\mathcal{E}}}$ to $\mathcal{E}$.

It will be convenient to extend vol by zero over the entire space $H^{1,1}(X, \mathbb{R})$. In this way, we have $\mathrm{vol}(\alpha) = 0$ unless α is big.

5.4 Alternative characterizations of the volume

Theorem 5.31. *Let X be a compact Kähler manifold and let α be a big class. Then*

$$\mathrm{vol}(\alpha) = \sup\left\{\int_X \langle T^n \rangle;\ T \text{ Kähler current in } \alpha \text{ with analytic singularities}\right\}.$$

Recall that when T has analytic singularities along a subvariety Z, then $\int_X \langle T^n \rangle = \int_{X\setminus Z} T^n$.

Proof. From Theorem 5.22, we have $\int_X \langle T^n \rangle \leqslant \mathrm{vol}(\alpha)$ for any positive current $T \in \alpha$ with analytic singularities. We have to show the (asymptotic) converse.

Let $T_{\min} = \theta + dd^c\varphi$ be a current with minimal singularities in α, and let $T' = \theta + dd^c\varphi' \in \alpha$ be a fixed Kähler current with analytic singularities; say $T' \geqslant \omega$ for some Kähler form ω. By Demailly's regularization theorem, on can find a sequence $(T'_k)_{k\geqslant 1}$ of currents with analytic singularities in α such that

- $T'_k = \theta + dd^c\varphi'_k$ with $\varphi'_k \downarrow \varphi$;
- $T'_k \geqslant -\varepsilon_k\omega$ for $\varepsilon_k \downarrow 0$.

In particular,

$$T_k := \varepsilon_k T' + (1-\varepsilon_k)T'_k \geqslant \varepsilon_k^2 \omega$$

is a Kähler current in α with analytic singularities. Let U be a Zariski open set where $\varphi', \varphi \in L^\infty_{\mathrm{loc}}(U)$. Since $\varphi'_k \downarrow \varphi$, $\varphi'_k \in L^\infty_{\mathrm{loc}}(U)$ and we have weak convergence $T'^p \wedge T'^{n-p}_k \to T'^p \wedge T^{n-p}_{\min}$ on U for any $0 \leqslant p \leqslant n$. It follows that $T_k^n \to T^n_{\min}$ weakly on U.

Now, choose $\delta > 0$ arbitrarily small and fix a relatively compact subset $V \Subset U$ such that $\int_V T_{\min}^n \geqslant \mathrm{vol}(\alpha) - \delta/2$. For k large enough, we have $\int_U T_k^n \geqslant \int_V T_{\min}^n - \delta/2 \geqslant \mathrm{vol}(\alpha) - \delta$. □

Theorem 5.32. *Let X be a compact Kähler manifold and let α be a big class. Then*

$$\mathrm{vol}(\alpha) = \sup_{\substack{\pi:\widetilde{X}\to X \\ \pi^*\alpha = \beta + E}} \int_{\widetilde{X}} \beta^n,$$

where the supremum is taking over all modifications $\pi : \widetilde{X} \to X$ such that β is Kähler and E is an effective divisor.

Proof. First of all, if π is as in the statement, then we have

$$\begin{aligned} \mathrm{vol}(\alpha) &= \mathrm{vol}(\pi^*\alpha) \\ &\geqslant \mathrm{vol}(\beta) \\ &= \int_X \beta^n \end{aligned}$$

where we have used successively Lemma 5.28, Proposition 5.24 and Corollary 5.25. So we have to show the asymptotic converse.

Let $\delta > 0$. By Theorem 5.31, there exists a Kähler current $T \in \alpha$ with analytic singularities such that $\int_X \langle T^n \rangle \geqslant \mathrm{vol}(\alpha) - \delta$. We choose a Kähler form ω such that $T \geqslant \omega$.

By Theorem 5.9 and the few lines past it, one can find a composition of smooth blow ups $\pi : \widetilde{X} \to X$ such that $\pi^*T = \theta + [D]$ where θ is semipositive and D is effective. In particular, we get

$$\int_{\widetilde{X}} \theta^n = \int_{\widetilde{X}} \langle (\pi^*T)^n \rangle = \int_X \langle T^n \rangle \geqslant \mathrm{vol}(\alpha) - \delta/2. \tag{5.19}$$

Moreover, we have $\pi^*T \geqslant \pi^*\omega$, hence $\theta \geqslant \pi^*\omega$. From Corollary 4.9, there exists an effective divisor $F = \sum a_i F_i$ on $\widetilde{X}$ supported on the exceptional locus of π such that $\pi^*[\omega] - \varepsilon F$ is a Kähler class. In particular, $[\theta] - \varepsilon F$ is a Kähler class (as a sum of a Kähler class and a semipositive class).

In conclusion, we have

$$\pi^*\alpha = \underbrace{([\theta] - \varepsilon F)}_{=:\beta} + \underbrace{(D + \varepsilon F)}_{=:E}.$$

and $\beta^n = \int_{\widetilde{X}} \theta^n + O(\varepsilon) \geqslant \mathrm{vol}(\alpha) - \delta/2 + O(\varepsilon)$ by (6.26). Choosing $\varepsilon > 0$ sufficiently small, one can ensure that $\beta^n \geqslant \mathrm{vol}(\alpha) - \delta$ as expected. □

Similar arguments show that we have the more general statement:

Theorem 5.33. *Let α be a big class. One can find a sequence $\mu_k : \widehat{X}_k \to X$ of blow ups with smooth center such that for any integer $1 \leqslant p \leqslant n$, we have*

$$\langle \alpha^p \rangle = \lim_{k\to+\infty} (\mu_k)_* \beta_k^p \quad \text{in} \quad H^{p,p}(X, \mathbb{R}).$$

where $\mu_k^*\alpha = \beta_k + E_k$ with β_k Kähler and E_k is an effective divisor.

Proof. Indeed, take the same sequence T_k of Kähler currents with analytic singularities approximating $T_{\min}$ and desingularize T_k as before by a map $\mu_k : \widehat{X}_k \to X$. We decompose $\mu_k^* T_k = \theta_k + [E_k]$ with $\beta_k = [\theta_k]$ Kähler.

In order to test the sought convergence, it is enough to do it against a class $[\gamma] \in H^{p,p}(X,\mathbb{R})$ which is represented by a closed, positive form γ. By the proof of Theoreom 5.22, we have

$$\begin{aligned}(\mu_k)_*\beta_k^p \cdot [\gamma] &= \int_{\widehat{X}_k} \theta_k^p \wedge \mu_k^*\gamma \\ &= \int_{\widehat{X}_k} \langle \mu_k^* T_k^p \rangle \wedge \mu_k^*\gamma \\ &= \int_X \langle T_k^p \rangle \wedge \gamma \\ &\leqslant \int_X \langle T_{\min}^p \rangle \wedge \gamma = \langle \alpha^p \rangle \cdot \gamma.\end{aligned}$$

We are left to proving that if ε is any positive number, we can find k large enough so that $\int_X \langle T_k^p \rangle \wedge \gamma \geqslant \int_X \langle T_{\min}^p \rangle \wedge \gamma - \varepsilon$. Let U be a Zariski open subset where $T_{\min}$ and T_k have locally bounded potentials. By week convergence $T_k^p \to T_{\min}^p$ on U, we can find $V \Subset V' \Subset U$ large enough and k big enough so that

$$\int_{V'} T_k^p \wedge \gamma \geqslant \int_V T_{\min}^p \wedge \gamma - \frac{\varepsilon}{2} \geqslant \int_U \langle T_{\min}^p \rangle \wedge \gamma - \varepsilon.$$

□

6 The duality theorem

6.1 The movable cone

Let X be a compact Kähler manifold. Recall that we have the Poincaré duality; i.e. the pairing

$$\begin{aligned} H^{1,1}(X,\mathbb{R}) \times H^{n-1,n-1}(X,\mathbb{R}) &\longrightarrow \mathbb{R} \\ (\alpha = [\omega], \gamma = [\omega']) &\longmapsto (\alpha \cdot \gamma) := \int_X \omega \wedge \omega' \end{aligned}$$

is non-degenerate.

We would like to characterize the dual cone $\mathcal{E}^\vee \subset H^{n-1,n-1}(X,\mathbb{R})$ of the pseudoeffective cone $\mathcal{E}$; that is

$$\mathcal{E}^\vee := \{\gamma \in H^{n-1,n-1}(X,\mathbb{R});\ (\alpha \cdot \gamma) \geqslant 0,\ \forall \alpha \in \mathcal{E}\}.$$

Clearly, $\mathcal{E}^\vee$ is a closed, convex cone inside $H^{n-1,n-1}(X,\mathbb{R})$.

For instance $\mathcal{E}^\vee$ contains all (non-negative combinations) of classes $[\omega^{n-1}]$ where ω is Kähler. We can actually find more elements in this cone:

Definition 6.1. Let X be a compact Kähler manifold and let $\gamma \in H^{n-1,n-1}(X,\mathbb{R})$. One says that γ is a movable class if there exists a bimeromorphic map $\mu : \widehat{X} \to X$ and a Kähler class $\beta \in H^{1,1}(\widehat{X},\mathbb{R})$ such that $\gamma = \mu_*(\beta^{n-1})$.

The movable cone $\mathcal{M} \subset H^{n-1,n-1}(X,\mathbb{R})$ is the closed, convex cone generated by all movable classes.

Example 6.2. For any big class $\alpha \in H^{1,1}(X,\mathbb{R})$, the class $\langle \alpha^{n-1} \rangle \in H^{n-1,n-1}(X,\mathbb{R})$ belongs to the movable cone. This is a consequence of Theorem 5.33 and the fact that $\mathcal{M}$ is closed.

Lemma 6.3. *We have* $\mathcal{M} \subset \mathcal{E}^\vee$.

Proof. It is sufficient to see that if α is psef and $\gamma = \mu_*(\beta^{n-1})$ is movable, then $(\alpha \cdot \gamma) \geqslant 0$. Let $\omega \in \beta$ be a Kähler form, and let $T \in \alpha$ be a positive current and let $\theta \in \alpha$ be a smooth representative. The current $\mu^* T \in \mu^* \alpha$ is positive hence

$$\begin{aligned}(\alpha \cdot \gamma) &= \int_X \theta \wedge \mu_*(\omega^{n-1}) \\ &= \int_{\widehat{X}} \mu^*\theta \wedge \omega^{n-1} \\ &= \int_{\widehat{X}} \mu^* T \wedge \omega^{n-1} \geqslant 0.\end{aligned}$$

The lemma follows. □

The culminating point of these lectures is the duality theorem stating that the reverse inclusion in Lemma 6.3 holds i.e. $\mathcal{M} = \mathcal{E}^\vee$; cf Theorem 6.10.

6.2 Holomorphic Morse inequalities and the orthogonality estimate

The proof of the holomorphic Morse inequalities in the (half)-transcendental case relies on the following result of Berman:

Theorem 6.4 (Berman)**.** *Let θ be a smooth, real, closed $(1,1)$-form on a compact Kähler manifold such that $\alpha = [\theta]$ is a Kähler class. Defined*

$$\phi := \sup\{\psi \leqslant 0;\ \psi \in \mathrm{PSH}(X,\theta)\},$$

and let $D = \{\phi = 0\}$. Then $\phi \in \mathrm{PSH}(X,\theta)$ is bounded, $dd^c\phi$ is also bounded and

$$(\theta + dd^c\phi)^n = 1_D \theta^n.$$

In particular, $\mathrm{vol}(\alpha) = \alpha^n = \int_X (\theta + dd^c\phi)^n = \int_D \theta^n$.

If $\theta \geqslant 0$, then of course $\phi \equiv 0$ and $D = X$. More generally, on the open set $X \setminus (\theta \geqslant 0)$, we can find local disks along which ϕ is strictly psh. In particular, $\phi < \sup_X \phi = 0$ at these points. In other words, $(\phi = 0) \subset (\theta \geqslant 0)$. This is consistent with the fact that $1_D\theta^n$ is a positive measure (as a Monge-Ampère of a psh function).

Using Bedford-Taylor solution of the Dirichlet problem, one can show that $(\theta + dd^c\phi)^n$ puts no mass on $(\phi < 0)$. Heuristically, if $\phi \in \mathcal{C}^2$, then $\theta + dd^c\phi$ is a form with continuous coefficients. If it is strictly positive at a point x and $\phi(x) < 0$, then the same holds in a the neighborhood U of x with coordinates (z_j) centered at x. If χ is a cutoff function, then $\phi + \varepsilon\chi\,|z|^2$ is still θ-psh and non-positive for $0 < \varepsilon \ll 1$, but it is strictly bigger than φ near x. This is absurd, hence $\theta + dd^c\phi > 0$ implies $\phi(x) = 0$. Moreover, since $\theta + dd^c\phi \geqslant 0$, the continuous volume form $(\theta + dd^c\phi)^n$ is strictly positive at x iff $\theta + dd^c\phi$ is strictly positive at x.

If f is a smooth function on X, one can consider the envelope

$$\phi_f := \sup\{\psi \leqslant f;\ \psi \in \mathrm{PSH}(X,\theta)\}.$$

Clearly $\phi_f - f = \sup\{\psi \leqslant 0;\ \psi \in \mathrm{PSH}(X, \theta + dd^c f)\}$, so that

$$\mathrm{vol}(\alpha) = \int_X (\theta + dd^c\phi_f)^n = \int_{D_f} (\theta + dd^c f)^n, \tag{6.20}$$

where $D_f = \{\phi_f = f\}$.

Theorem 6.5 (Witt-Nyström). *Let X be a projective manifold, and let $\alpha, \beta \in H^{1,1}(X,\mathbb{R})$ be two nef classes where $\beta \in \mathrm{NS}_{\mathbb{R}}(X)$. Then,*

$$\mathrm{vol}(\alpha - \beta) \geqslant (\alpha^n) - \sum_{k=1}^{n} \binom{n}{k} (\alpha^{n-k} \cdot \beta^k).$$

Remark 6.6. Boucksom proved that Theorem (6.5) actually implies the stronger statement that for any nef classes α, β (without assuming β rational), we have the improved inequality $\mathrm{vol}(\alpha - \beta) \geqslant (\alpha^n) - n(\alpha^{n-1} \cdot \beta)$.

Remark 6.7. When both α and β are rational (i.e. they are the Chern class of nef $\mathbb{Q}$-divisors D and E respectively), the the stronger inequality in the remark above is a rather simple consequence of Riemann-Roch theorem. Indeed, an easy continuity argument shows that one can assume that D and E are very ample Cartier divisors. Next, fix a (large) integer m and pick $E_1, \ldots, E_m \in |E|$ general members so that $H^0(X, m(D-E)) \simeq H^0(X, mD - \sum_{i=1}^m E_i)$ corresponds to sections of mD vanishing along each E_i. Use the exact sequence

$$0 \longrightarrow H^0(X, m(D-E)) \longrightarrow H^0(X, mD) \longrightarrow \bigoplus_{i=1}^{m} H^0(E_i, mD|_{E_i})$$

and asymptotic Riemann-Roch for nef line bundles to obtain

$$\begin{aligned} h^0(X, m(D-E)) &\geqslant h^0(X, mD) - \sum_{i=1}^m h^0(E_i, mD|_{E_i}) \\ &= \frac{(D^n)}{n!} m^n - \sum_{i=1}^m \frac{(D|_{E_i}^{n-1})}{(n-1)!} m^{n-1} + O(m^{n-1}) \\ &= \frac{m^n}{n!} \Big((D^n) - \frac{1}{m} \sum_{i=1}^m n(D^{n-1} \cdot E_i) + O(m^{-1}) \Big) \\ &= \frac{m^n}{n!} \Big((D^n) - n(D^{n-1} \cdot E) + O(m^{-1}) \Big) \end{aligned}$$

hence the expected inequality follows.

Proof. We proceed in several steps.

Step 1. *Reduction step.*
First of all, one can assume that the RHS is positive, otherwise there is nothing to prove. In particular, $(\alpha^n) > 0$ and α is big. Next, assume that we can show the inequality when $\alpha - \beta$ is big. Then, for general α, β, we can consider the positive number $t_0 = \sup\{t \in [0,1]; \alpha - t\beta \text{ is big}\}$. We have

$$\begin{aligned} \mathrm{vol}(\alpha - t_0\beta) = \lim_{t \to t_0^-} \mathrm{vol}(\alpha - t\beta) &\geqslant (\alpha^n) - \sum_{k=1}^{n} \binom{n}{k} t^k (\alpha^{n-k} \cdot \beta^k) \\ &\geqslant (\alpha^n) - \sum_{k=1}^{n} \binom{n}{k} (\alpha^{n-k} \cdot \beta^k) > 0 \end{aligned}$$

hence $\alpha - t_0\beta$ is big and $t_0 = 1$. So we can assume that $\alpha - \beta$ is big from now on.

Next, up to replacing α by $\alpha + \varepsilon\sigma$ (and similarly for β) for an integral Kähler class σ, the continuity of intersection numbers allows us to assume that α and β are Kähler. By continuity of the volume, one can assume that β is a rational class. Finally, by homogeneity of the formula, one can assume that β is integral. In summary, α is a Kähler class and $\beta = c_1(L)$ for some positive line bundle $L \to X$ with a non-zero section s.

We choose Kähler representatives $\theta \in \alpha$, $\omega \in \beta$. There exists an hermitian metric on L such that $\Theta_h(L) = \omega$, hence

$$g := \log |s|_h^2$$

is an ω-psh function. We can rescale h so that $\sup_X g = 0$. Setting $Y := (s = 0)$, we get $dd^c g = [Y] - \omega$.

Step 2. *Construction of a limit envelope.*
We would like to define the envelope of θ-psh that are dominated by g, but we can't do it directly since g is singular. Instead, we consider for any (large) number $R > 0$ a (regularized) maximum

$$g_R := \max(g, -R).$$

Clearly, $g_R \in \mathrm{PSH}(X, \omega)$ and it decreases to g when $R \to +\infty$. We introduce

$$\phi_R := \{\psi \leqslant g_R; \psi \in \mathrm{PSH}(X, \theta)\}$$

and set $D_R = (\phi_R = g_R)$. The function ϕ_R is bounded (by $-R$) hence it has minimal singularities, and

$$(\alpha^n) = \mathrm{vol}(\alpha) = \int_X (\theta + dd^c\phi_R)^n. \tag{6.21}$$

Furthermore, it follows from (6.20) that

$$(\theta + dd^c\phi_R)^n = 1_{D_R}(\theta + dd^c g_R)^n. \tag{6.22}$$

Let $\phi_\infty := \lim_{R \to +\infty} \phi_R$, which exists since ϕ_R decreases. It is a θ-psh function unless it is identically $-\infty$. Let us show that it is not the case. Since $\alpha - \beta$ is big, we can consider a function $v \in \mathrm{PSH}(X, \theta - \omega)$ with minimal singularities, normalized by $\sup_X v = 0$. Clearly, $\theta + dd^c(v+g) = (\theta - \omega + dd^c v) + (\omega + dd^c g) \geqslant 0$ hence $v + g \in \mathrm{PSH}(X, \theta)$ and it satisfies $v + g \leqslant g \leqslant g_R$ for all R. This implies that $v + g \leqslant \phi_R$, hence $\phi_\infty \not\equiv -\infty$.

Next, we claim that $\phi_\infty - g \in \mathrm{PSH}(X, \theta - \omega)$ has minimal singularities. Indeed, $\phi_\infty - g \leqslant 0$ by construction, and $\theta - \omega + dd^c(\phi_\infty - g) = (\theta + dd^c \phi_\infty) \geqslant 0$ on $X \setminus Y$ hence $\phi_\infty - g \in \mathrm{PSH}(X, \theta - \omega)$. Since $\phi_\infty - g \geqslant v$, it has minimal singularities too.

Finally, let us prove that

$$\mathrm{vol}(\alpha - \beta) = \int_X \langle (\theta + dd^c \phi_\infty)^n \rangle. \tag{6.23}$$

Let $Z = X \setminus E_{\mathrm{nK}}(\alpha - \beta)$. Since $\phi_\infty - g$ has minimal singularities, we have $\mathrm{vol}(\alpha - \beta) = \int_X \langle (\theta - \omega + dd^c(\phi_\infty - g))^n \rangle$. On $X \setminus (Y \cup Z)$, we have $\theta - \omega + dd^c(\phi_\infty - g) = \theta + dd^c \phi_\infty$ and all the relevant functions are locally bounded there. Since the non-pluripolar Monge-Ampère puts no mass on $Y \cup Z$, we have

$$\begin{aligned}\int_X \langle (\theta - \omega + dd^c(\phi_\infty - g))^n \rangle &= \int_{X \setminus (Y \cup Z)} (\theta - \omega + dd^c(\phi_\infty - g))^n \\ &= \int_{X \setminus (Y \cup Z)} (\theta + dd^c \phi_\infty)^n \\ &= \int_X \langle (\theta + dd^c \phi_\infty)^n \rangle.\end{aligned}$$

Step 3. *A Monge-Ampère mass computation.*
Choose a large open set $U \Subset X \setminus (Y \cup Z)$. Since $\phi_R \downarrow \phi_\infty$, we have weak convergence $(\theta + dd^c \phi_R)^n \to (\theta + dd^c \phi_\infty)^n$ on U. In particular, we get from (6.23) that

$$\mathrm{vol}(\alpha - \beta) \geqslant \int_{\overline{U}} (\theta + dd^c \phi_\infty)^n \geqslant \lim_{R \to +\infty} \int_U (\theta + dd^c \phi_R)^n.$$

In combination with (6.21)-(6.22), we thus get

$$\mathrm{vol}(\alpha - \beta) \geqslant (\alpha^n) - \lim_{R \to +\infty} \int_{D_R \cap U^c} (\theta + dd^c g_R)^n. \tag{6.24}$$

Although the function g_R is not θ-psh, it is dd^c-bounded and satisfies

$$1_{D_R} (\theta + dd^c g_R)^n \leqslant (\theta + (\omega + dd^c g_R))^n = \sum_{k=0}^{n} \binom{n}{k} \theta^{n-k} \wedge (\omega + dd^c g_R)^k. \tag{6.25}$$

For U large enough, $\int_{U^c} \theta^n$ gets arbitrary small, while for the other terms $(k < n)$, we have

$$\int_{U^c} \theta^{n-k} \wedge (\omega + dd^c g_R)^k \leqslant \int_X \theta^{n-k} \wedge (\omega + dd^c g_R)^k = (\alpha^{n-k} \cdot \beta^k)$$

by Bedford-Taylor, since g_R is bounded. The conclusion now follows from (6.24) by integrating (6.25). □

The above volume estimate, often referred to as a holomorphic Morse inequality, has the following striking application, due to Boucksom-Demailly-Păun-Peternell in the rational case and Witt-Nyström in general, which is the key for the duality theorem as we will see later.

Theorem 6.8 (Boucksom-Demailly-Peternell-Păun, Witt-Nyström)**.** *Let X be a projective manifold and let α be a big class. Then*

$$\mathrm{vol}(\alpha) = (\alpha \cdot \langle \alpha^{n-1} \rangle).$$

Proof. Thanks to Theorem 5.33, we can find a sequence of modifications $\mu_k : \widehat{X}_k \to X$ such that

$$\mu_k^* \alpha = \alpha_k + E_k,$$

where α_k is Kähler, E_k is effective, and

$$\mathrm{vol}(\alpha) = \lim(\alpha_k^n); \quad \text{and} \quad \langle \alpha^{n-1} \rangle = \lim(\mu_k)_* \alpha_k^{n-1}. \tag{6.26}$$

In particular,

$$\begin{aligned} \mathrm{vol}(\alpha) - (\alpha \cdot \langle \alpha^{n-1} \rangle) &= \lim_{k \to +\infty} (\alpha_k^n - (\mu_k^* \alpha \cdot \alpha_k^{n-1}) \\ &= \lim_{k \to +\infty} (E_k \cdot \alpha_k^{n-1}). \end{aligned} \tag{6.27}$$

Let σ be an integral Kähler class (whose existence is guaranteed by the projectivity of X); up to scaling, one can assume that $\sigma - \alpha$ is a Kähler class too. Moreover, the class

$$\mu_k^* \sigma - E_k = \mu_k^*(\sigma - \alpha) + \alpha_k$$

is both integral and Kähler and we can decompose for any $t \in [0, 1]$

$$\alpha_k + tE_k = (\alpha_k + t\mu_k^* \sigma) - t(\mu_k^* \sigma - E_k)$$

as a difference of two Kähler classe. Applying Theorem 6.5, we get the following

$$\begin{aligned} \mathrm{vol}(\alpha) = \mathrm{vol}(\alpha_k + E_k) \geqslant & \mathrm{vol}(\alpha_k + tE_k) \\ \geqslant & (\alpha_k + t\mu_k^* \sigma)^n - nt(\alpha_k + t\mu_k^* \sigma)^{n-1} \cdot (\mu_k^* \sigma - E_k) \\ & - \sum_{k=2}^{n} \binom{n}{k} t^k (\alpha_k + t\mu_k^* \sigma)^{n-k} \cdot (\mu_k^* \sigma - E_k)^k. \end{aligned}$$

Since $\alpha_k \leqslant \mu_k^* \alpha \leqslant \mu_k^* \sigma$ and $\mu_k^* \sigma - E_k \leqslant \mu_k^* \sigma$, Lemma 6.9 below shows that $(\alpha_k + t\mu_k^* \sigma)^{n-k} \cdot (\mu_k^* \sigma - E_k)^k \leqslant 2^{n-k}(\sigma^n)$. In particular, we find a constant $C > 0$ such that

$$\mathrm{vol}(\alpha) \geqslant (\alpha_k^n) + nt(\alpha_k^{n-1} \cdot E_k) - Ct^2.$$

Take $t = 1$, this yields $(\alpha_k^{n-1} \cdot E_k) \leqslant C'$ for some constant C'. Up to increasing C, one can assume that $C' \leqslant \frac{C}{2n}$. Now, take $t = \frac{n(\alpha_k^{n-1} \cdot E_k)}{2C} \in [0, 1]$; it yields

$$(\alpha_k^{n-1} \cdot E_k) \leqslant \frac{4C}{n^2} (\mathrm{vol}(\alpha) - (\alpha_k^n)).$$

Combining this with (6.26) and (6.27), the corollary is proved. □

In the course of the proof of Corollary 6.8, we have used the lemma below.

Lemma 6.9. *Let X be a compact Kähler manifold and let $\alpha_1, \dots, \alpha_n$ (resp. $\alpha'_1, \dots, \alpha'_n$) be nef classes such that $\alpha_i \leqslant \alpha'_i$ in the sense that $\alpha'_i - \alpha_i$ is psef. Then*

$$\alpha_1 \cdot \ldots \cdot \alpha_n \leqslant \alpha'_1 \cdot \ldots \cdot \alpha'_n.$$

Proof. We do it one class at a time:

$$\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n \leqslant \alpha'_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha'_n,$$

since the classes $\alpha_2, \dots, \alpha_n$ are nef, and we iterate. □

6.3 The duality theorem

Theorem 6.10. *Let X be a projective manifold. The pseudoeffective cone is the dual cone of the movable cone; that is $\mathcal{E} = \mathcal{M}^\vee$.*

Proof. By Lemma 6.3, we have $\mathcal{M}^\vee \supset \mathcal{E}^{\vee\vee} = \mathcal{E}$, so it is enough to show the reverse inclusion $\mathcal{M}^\vee \subset \mathcal{E}$. Since $\mathcal{E}$ is closed, it is enough to show that $\mathring{\mathcal{M}}^\vee \subset \mathring{\mathcal{E}}$. And since $\mathcal{M}^\vee$ contains $\mathcal{E}$, it comes down to showing that $\mathring{\mathcal{M}}^\vee \cap \mathcal{E} \subset \mathring{\mathcal{E}}$.

Let σ be a Kähler class and let $\alpha \in \mathcal{E}$ be in the interior of $\mathcal{M}^\vee$. Clearly $\alpha + \varepsilon\sigma$ is big for any $\varepsilon > 0$, and moreover, $\alpha - \varepsilon_0\sigma \in \mathcal{M}^\vee$ for some $\varepsilon_0 > 0$. Thanks to Example 6.2, it follows that for any big class β, we have

$$\alpha \cdot \langle \beta^{n-1} \rangle \geqslant \varepsilon_0 \sigma \cdot \langle \beta^{n-1} \rangle$$

Using successively Theorem 6.8, the property above, Lemma 5.27 and Theorem 5.29, we find

$$\begin{aligned} \mathrm{vol}(\alpha + \varepsilon\sigma) &= (\alpha + \varepsilon\sigma) \cdot \langle (\alpha + \varepsilon\sigma)^{n-1} \rangle \\ &\geqslant \varepsilon_0 \left(\sigma \cdot \langle (\alpha + \varepsilon\sigma)^{n-1} \rangle \right) \\ &= \varepsilon_0 \left\langle \sigma \cdot (\alpha + \varepsilon\sigma)^{n-1} \right\rangle \\ &\geqslant \varepsilon_0 \langle \sigma^n \rangle^{\frac{1}{n}} \,\mathrm{vol}(\alpha + \varepsilon\sigma)^{\frac{n-1}{n}}. \end{aligned}$$

At the end of the day, one finds $\mathrm{vol}(\alpha + \varepsilon\sigma) \geqslant \varepsilon_0^n \mathrm{vol}(\sigma)$, and by Theorem 5.30, we conclude that α is big. □

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