GEOMETRIC APPLICATIONS OF KÄHLER-EINSTEIN METRICS

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ABSTRACT. These lecture notes aim to describe a few striking and by now classical applications of the seminal theorems of Aubin-Yau and Yau on the existence of Kähler-Einstein metrics on compact Kähler manifolds with zero or negative first Chern class.

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1. RIEMANN SURFACES: A BRIEF TOUR

A Riemann surface is a complex manifold X of complex dimension one.

Example 1.1. An open subset $\Omega \subset \mathbb{C}$, or the projective line $\mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\} \overset{\text{top}}{=} S^2$.

The following theorem is due to Riemann, Poincaré and Koebe.

Theorem 1.2. *Let X be a simply connected Riemann surface. Then X is biholomorphic either to $\mathbb{P}^1, \mathbb{C}$ or $\mathbb{B} = \{z \in \mathbb{C}; |z| < 1\}$.*

As a corollary, if X is a compact Riemann surface, then one has the following trichotomy

$\widetilde{X}$	$\mathbb{P}^1$	$\mathbb{C}$	$\mathbb{B}$
X	$\mathbb{P}^1$	torus $\mathbb{C}/\Lambda$, Λ lattice	$\mathbb{B}/\Gamma$
$e(X)$	> 0	0	< 0
$\pi_1(X)$	$\{1\}$	$\Lambda \simeq \mathbb{Z}^2$	Γ infinite non-abelian
$\mathrm{Aut}^\circ(\widetilde{X})$	$PGL(2,\mathbb{C})$	$\mathrm{Aff}(\mathbb{C}) = \mathbb{C} \rtimes \mathbb{C}^*$	$\{1\}$
$\mathrm{Aut}^\circ(X)$	$PGL(2,\mathbb{C})$	$\mathbb{C}/\Lambda$	$PU(1,1) \simeq PSL(2,\mathbb{R})$

In the diagram above, $e(X)$ is the topological Euler characteristic. It coincides with $\int c_1(X)$, or also $2-2g$ where $g = h^1(X, \mathcal{O}_X) = h^0(X, \Omega^1_X)$ is the genus of X.

Each of the three model spaces admits a distinguished complete metric

$$\omega_{\mathbb{P}^1} = i\partial\bar\partial \log(1+|z|^2) = \frac{idz\wedge d\bar z}{(1+|z|^2)^2} \quad \text{on} \quad \mathbb{C} \subset \mathbb{P}^1$$
$$\omega_{\mathbb{C}} = i\partial\bar\partial |z|^2 = idz\wedge d\bar z$$
$$\omega_{\mathbb{B}} = -i\partial\bar\partial \log(1-|z|^2) = \frac{idz\wedge d\bar z}{(1-|z|^2)^2}$$

satisfying the property that their Gauss curvature is identically 1 (resp. 0, -1). Recall that if ω is a positive $(1,1)$-form written in a chart $\omega \underset{\text{loc}}{=} f(z) idz\wedge d\bar z$, then

$$K_\omega = -\frac{1}{2}\frac{\Delta f}{f}$$

where $\Delta = \frac{\partial^2}{\partial z \partial \bar z}$ is the euclidean Laplacian. This expression is easily seen to be independent of the chosen chart.

The three metrics $\omega_{\widetilde{X}}$ above are invariant under $\pi_1(X)$ hence descend to a metric ω_X on X with constant Gauss curvature.

The goal of these lectures is to touch upon the question of what happens to higher dimensional compact complex manifolds X? More precisely, here is a list of vague questions that we will try to address through the lens of Kähler-Einstein metrics.

(1) Can we classify all possible universal covers $\widetilde{X}$? Are there restrictions to what groups $\pi_1(X)$ can occur?
(2) Does X carry one or several "canonical" metrics? If so, what governs the existence of such metrics and what are some topological or geometric implications?

(3) Are there numerical (or finer) invariants that can detect special geometric behavior?

2. Curvature of Kähler manifolds and metric uniformization

2.1. **Kähler metrics.** Let X be a complex manifold of dimension n. We denote by T_X (resp. $\Omega_X = T_X^*$) the holomorphic tangent bundle (resp. cotangent bundle). The canonical bundle K_X of X is the line bundle $K_X := \det(\Omega_X) = \Lambda^n \Omega_X$.

For integers $0 \leqslant p, q \leqslant n$, we denote by $\Omega_X^{p,q} \subset \Omega_{X,\mathbb{C}}^{p+q}$ the complex bundle of differential forms of bi-degree (p,q). That is, if we have a local system of coordinates $z_1, \ldots, z_n$, then T_X is generated by the holomorphic vector fields $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ while Ω_X is generated by $dz_1, \ldots, dz_n$. As for $\Omega_X^{p,q}$, it is generated by the forms of the type $dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$ for all possible sets of indices $1 \leqslant i_1 < i_2 < \ldots < i_p$ and $1 \leqslant j_1 < j_2 < \ldots < j_q$.

Definition 2.1. (Kähler metric) A Kähler form is a smooth $(1,1)$-form $\omega \in C^\infty(X, \Omega_X^{1,1})$ such that

(1) ω is positive, i.e. $-i\omega_x(u, \bar{u}) > 0$ for any $x \in X$ and $u \in T_{X,x}^{1,0} \setminus \{0\}$.
(2) ω is closed, i.e. $d\omega = 0$.

In local coordinates, one can write

$$\omega = \sqrt{-1} \sum_{1 \leqslant i,j \leqslant n} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j,$$

where $(g_{i\bar{j}})_{ij}$ is hermitian definite positive. In particular, $\omega = \overline{\omega}$ is real.

Remark 2.2. The positivity condition may be confusing at first sight. Recall that one can canonically attach to a real $(1,1)$-form $\omega \in \Omega_{X,\mathbb{R}}^{1,1}$ a hermitian form h on T_X by $h(u,v) = \omega(u, Jv) - i\omega(u,v)$ for $u, v \in T_X$. Writing $u = \eta + \bar{\eta}$ (with $\eta = \frac{1}{2}(u - iJu) \in T_X^{1,0}$, cf (2.3) below), one finds

$$h(u,u) = -2i\omega(\eta, \bar{\eta})$$

hence the convention so that h is hermitian definite positive when ω is Kähler.

As a consequence of the closeness of ω, one can attach to it its de Rham cohomology class

$$[\omega] \in H^2(X, \mathbb{R}).$$

Definition 2.3. (Kähler class) A cohomology class $\alpha \in H^2(X, R)$ is Kähler if there exists a Kähler metric ω such that $[\omega] = \alpha$, i.e. ω represents α. One will write for short $\alpha > 0$ (resp. $\alpha < 0$ if α (resp. $-\alpha$) is a Kähler class

2.2. **What is curvature?** If ω is a Kähler metric, the condition $d\omega = 0$ allows to find near each point of x a set of holomorphic coordinates $(z_1, \ldots, z_n)$ centered at x such that

$$\omega = \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j; \quad g_{i\bar{j}}(z) = \delta_{ij} + O(|z|^2).$$

In other words, ω osculates to the flat metric up to order 1 (note that this condition for an arbitrary hermitian metric implies $d\omega = 0$). However, it is not possible to improve the order of contact in general but one can refine the expansion and write

$$g_{i\bar{j}}(z) = \delta_{ij} + \sum_{i,j,k,\ell} c_{i\bar{j}k\bar{\ell}} z_k \bar{z}_\ell + O(|z|^3). \tag{2.1}$$

Of course all of this is heavily dependent on the choice of coordinates (and there exist many choices of coordinates for which ω will have that shape). It turns out, however, that the contravariant order 4 tensor $c_{i\bar{j}k\bar{\ell}}$ is intrinsically defined. In other words, if $u, v, w, t \in T_{X,x}$, one can pick a special set or coordinates (z_i) as before and decompose $u = \sum_i u_i \frac{\partial}{\partial z_i}$ (and similarly for v, w, t). Then the number

$$R(u, v, w, t) := -\sum_{i,j,k,\ell} c_{i\bar{j}k\bar{\ell}}\, u_i \overline{v}_j t_k \overline{w}_\ell \tag{2.2}$$

does not depend on the choice of particular coordinates, but rather just depends on (the 2-jet of) ω.

Let us now recall how one can formalize this construction. If ω is a Kähler metric, it induces a complex symmetric tensor field g (i.e. a section of $S^2 T^*_{X,\mathbb{C}}$) by $g(u, v) = \omega(u, Jv)$ where $J \in C^\infty(X, \mathrm{End}(T_{X,\mathbb{C}}))$ is the $\mathbb{C}$-linear extension of the complex structure on X. In particular, $g(u, v) = 0$ if $u, v \in T^{1,0}_X$ or $u, v \in T^{0,1}_X$. In turn, g induces a hermitian inner product h on $T^{1,0}_X$ by $h(u, v) = g(u, \bar{v})$. The natural map $j : T_X \to T_{X,\mathbb{C}}$ given by $j(v) = v \otimes 1$ has image $T_{X,\mathbb{R}} = \{w \in T_{X,\mathbb{C}};\ \overline{w} = w\} = \{\eta + \bar{\eta};\ \eta \in T^{1,0}_X\}$. The composition of j and the projection $T_{X,\mathbb{C}} \to T^{1,0}_X$ yields an isomorphism of complex bundles

$$\begin{aligned} T_X &\to T^{1,0}_X \\ u &\mapsto \frac{1}{2}(u - iJu). \end{aligned} \tag{2.3}$$

Finally, if V is a complex vector space of dimension n, pick a basis $\mathcal{B} := (e_k)$ of V so that (Je_k) is an $\mathbb{R}$-basis of V. Then $\mathcal{B}_{\mathbb{C}} := (e_k - iJe_k)_k \cup (e_k + iJe_k)_k$ forms a $\mathbb{C}$ basis of $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ adapted to the decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. If $f : V \to V$ is a complex linear endomorphism whose matrix is $(f_{k\ell})$ with respect to $\mathcal{B}$, then the induced complex endomorphism $f^{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ has matrix $\begin{pmatrix} f_{k\ell} & 0 \\ 0 & f_{k\ell} \end{pmatrix}$ with respect to $\mathcal{B}_{\mathbb{C}}$.

One can then associate to (T_X, h) its Chern connection ∇^h with Chern curvature $\Theta(T_X, h) = (\nabla^h)^2$. The connection ∇^h extends $\mathbb{C}$-linearly to $T_{X,\mathbb{C}}$; it is straighforward to check that $\nabla^h J = 0$; in particular ∇^h preserves the decomposition $T^{1,0}_X \oplus T^{0,1}_X$.

We set

$$\Theta_\omega := \frac{i}{2\pi}\Theta(T_X, h) \in C^\infty(X, \mathrm{End}(T_X) \otimes \Omega^{1,1}_X).$$

which we can extend $\mathbb{C}$-linearly to a section $\Theta_\omega^{\mathbb{C}}$ of $\mathrm{End}(T_{X,\mathbb{C}}) \otimes \Omega_X^{1,1}$. In particular, for any $u, v \in T_{X,\mathbb{C}}$, the endormorphism $\Theta_\omega^{\mathbb{C}}(u,v)$ of $T_{X,\mathbb{C}}$ preserves $T_X^{1,0}$ and $T_X^{0,1}$.

Locally, one can represent Θ_ω as a $n \times n$ matrix whose entries are real $(1,1)$-forms. More concretely, if $u, v, w, t \in T_{X,x}^{\mathbb{C}}$, then one can form the quantity

$$R(u,v,w,t) := g(\underbrace{\Theta_\omega^{\mathbb{C}}(u,v)}_{\in \mathrm{End}(T_{X,x}^{\mathbb{C}})} t, w) \in \mathbb{C}$$

A priori $R \in C^\infty(X, (\Omega_{X,\mathbb{C}}^1)^{\otimes 4})$ but its naturally symmetries allow it so see it as a section of the subbundle $S^2\Omega_X^{1,1} \subset (\Omega_{X,\mathbb{C}}^1)^{\otimes 4}$, i.e. $R(u,v,w,t)$ only depends on the image of the projection $T_{X,\mathbb{C}}^{\otimes 4} \ni u \otimes v \otimes w \otimes t \mapsto (u \wedge v) \otimes (w \wedge t) \in S^2\Omega_X^{1,1}$. For instance, one sees that $R(u,v,w,t) = 0$ if w, t have the same type since $\Theta_\omega^{\mathbb{C}}(u,v)$ preserves the type and $g(a,b) = 0$ if a, b have the same type. In local coordinates, one can write

$$\Theta_\omega = \sqrt{-1} \sum_{i,j,k,l} R_{i\ k\bar{\ell}}^{\ j} \Big(\frac{\partial}{\partial z_i}\Big)^* \otimes \frac{\partial}{\partial z_j} \otimes dz_k \wedge d\bar{z}_\ell,$$

$$R = \sqrt{-1} \sum_{i,j,k,l} R_{i\bar{j}k\bar{\ell}}\, dz_i \wedge d\bar{z}_j \otimes dz_k \wedge d\bar{z}_\ell$$

with the usual $R_{i\bar{j}k\bar{\ell}} = \sum_p g_{p\bar{j}} R_{i\ k\bar{\ell}}^{\ p}$. One can then check that R coincides with the expression (2.2) given above. The induced tensor representation for $\Theta_\omega^{\mathbb{C}}$ is obtained from the previous considerations.

Definition 2.4. (Ricci form) If ω is a Kähler metric, we define

$$\mathrm{Ric}(\omega) := \mathrm{tr}_{\mathrm{End}}\Theta_\omega = \mathrm{tr}_\omega R \in C^\infty(X, \Omega_X^{1,1}).$$

In coordinates, we get $\mathrm{Ric}(\omega)_{k\bar{\ell}} =: R_{k\bar{\ell}} = R^i_{ik\bar{\ell}} = g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}$ with the Einstein convention that we implicitly sum over i (resp. i, j). The identity of the two traces follows from the symmetry $R_{i\bar{j}k\bar{\ell}} = R_{k\bar{\ell}i\bar{j}}$ holding in normal coordinates, i.e. satisfying (2.1).

Since ω is Kähler, $\mathrm{Ric}(\omega)$ is a closed form and one has the formula

$$\mathrm{Ric}(\omega) = -i\partial\bar{\partial} \log \omega^n$$

or $R_{j\bar{k}} = -\sqrt{-1}\partial^2_{j\bar{k}} \log \det(g_{i\bar{j}})$. Moreover, it corresponds to the usual Riemannian Ricci tensor under the suitable identifications.

Definition 2.5. (Scalar curvature) If ω is a Kähler metric, we define $s(\omega) := \mathrm{tr}_\omega \mathrm{Ric}(\omega) \in C^\infty(X)$, or in coordinates $g^{k\bar{\ell}} R_{k\bar{\ell}}$.

2.3. **Holomorphic sectional curvature.** This section is devoting to studying the holomorphic analog of the Riemannian sectional curvature. The definition is as follows.

Definition 2.6. (Holomorphic sectional curvature) If ω is a Kähler metric, we define for $u \in T_{X,x}^{1,0} \setminus \{0\}$ the quantity

$$\mathrm{HSC}_\omega(u) := \frac{1}{|u|^4} R(u, \bar{u}, u, \bar{u}).$$

One can prove that $\mathrm{HSC}_\omega(u)$ corresponds to the supremum of the quantities $K_{f^*\omega}(0)$ where f ranges among all holomorphic disks $f : \mathbb{D} \to X$ such that $f(0) = x$ and $f'(0) = u$.

If $\xi \in T_{X,x}$ (seen as a real vector), then if one sets $u := \frac{1}{2}(\xi - iJ\xi) \in T^{1,0}_{X,x}$, one finds

$$R(u, \bar{u}, u, \bar{u}) = \frac{1}{4} R(\xi, J\xi, \xi, J\xi).$$

We say that ω has constant holomorphic sectional curvature if there exists $\kappa \in \mathbb{R}$ such that $\mathrm{HSC}_\omega(u) = \kappa$ for any $x \in X$ and $u \in T_{X,x} \setminus \{0\}$. Arguments from linear algebra (see [KN69, IX Proposition 7.3]) show that this implies for $u, v, w \in T^{1,0}_X$

$$R(u, \bar{u}, v, \bar{v}) = \frac{\kappa}{2}(|u|^2|v|^2 + |g(u, \bar{v})|^2),$$

or in tensor notations

$$R_{i\bar{j}k\bar{\ell}} = \frac{\kappa}{2}(g_{i\bar{j}} g_{k\bar{\ell}} + g_{i\bar{\ell}} g_{k\bar{j}}). \tag{2.4}$$

In particular, ω is Kähler-Einstein, i.e.

$$\mathrm{Ric}(\omega) = \frac{1}{2}\kappa(n+1)\omega.$$

In terms of the tensor Θ_ω, we get for any $u, v, w \in T^{1,0}_X$

$$\Theta_\omega(u, v)w = \frac{\kappa}{2}(g(u, \bar{v})w + g(w, \bar{v})u).$$

Let us give one more intrinsic interpretation of what (2.4) means. Set $E := T^{1,0}_X$. Using ω, one can the tensor R (living in $S^2\Lambda^{1,1}E^*$) into an endomorphism $\widehat{R} : \Lambda^{1,1}E^* \to \Lambda^{1,1}E^*$. In terms of tensors, we have $\widehat{R}_{i\bar{j}}^{\ k\bar{\ell}} = g^{\alpha\bar{\ell}} g^{k\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}}$. One has two distinguished elements in $\mathrm{End}(\Lambda^{1,1}E^*)$, namely the identity Id and

$$\widehat{\omega} : \eta \mapsto \mathrm{tr}_\omega(\eta)\, \omega.$$

In coordinates, the identity is given by $\delta_{ik}\delta_{j\ell}$ and $\widehat{\omega}$ is given by $g^{k\bar{\ell}} g_{i\bar{j}}$. Reversing the "hat" operation, these two endomorphisms correspond respectively to the tensors in $S^2\Lambda^{1,1}E^*$ given by

$$g_{k\bar{\beta}} g_{\alpha\bar{\ell}} \delta_{ik}\delta_{j\ell} = g_{i\bar{\beta}} g_{\alpha\bar{j}}, \quad \text{and} \quad g_{k\bar{\beta}} g_{\alpha\bar{\ell}} g^{k\bar{\ell}} g_{i\bar{j}} = g_{\alpha\bar{\beta}} g_{i\bar{j}}. \tag{2.5}$$

Given an element $f = f_{i\bar{j}}^{\ k\bar{\ell}}$ in $S^2\Lambda^{1,1}E^*$ we can trace it out in two ways, namely

$$\mathrm{tr}_{\mathrm{End}}(f) = f_{i\bar{j}}^{\ i\bar{j}}, \quad \text{and} \quad \mathrm{tr}_\omega(f) = g^{i\bar{j}} g_{k\bar{\ell}} f_{i\bar{j}}^{\ k\bar{\ell}}.$$

It is elementary to check that

$$\mathrm{tr}_{\mathrm{End}}(\mathrm{Id}) = n^2, \mathrm{tr}_\omega(\mathrm{Id}) = n$$
$$\mathrm{tr}_{\mathrm{End}}(\widehat{\omega}) = n, \mathrm{tr}_\omega(\widehat{\omega}) = n^2$$
$$\mathrm{tr}_{\mathrm{End}}(\widehat{R}) = \mathrm{tr}_\omega(\widehat{R}) = s(\omega)$$

where s is the scalar curvature (the third identity uses the symmetry $R_{i\bar{j}k\bar{\ell}} = R_{i\bar{\ell}k\bar{j}}$). In particular, the tensor

$$\widehat{R}^{\circ} := \widehat{R} - \frac{s}{n(n+1)}(\mathrm{Id} + \widehat{\omega}) \tag{2.6}$$

is tracefree (wrt both traces). Given (2.5), one sees that ω has constant holomorphic sectional curvature (say at a point) iff $\widehat{R}^{\circ} = 0$.

2.4. **Metric uniformization.** The three simply connected Riemann surfaces and their respective complete constant curvature metrics admit analogs to arbitrary dimensions. More precisely, one can define

$$\begin{aligned}
\omega_{\mathbb{P}^n} &= i\partial\bar{\partial}\log(1+|z|^2) \quad \text{on} \quad \mathbb{C}^n \subset \mathbb{P}^n \\
\omega_{\mathbb{C}^n} &= i\partial\bar{\partial}|z|^2 = \sqrt{-1}\sum_i dz_i \wedge d\bar{z}_i \\
\omega_{\mathbb{B}^n} &= -i\partial\bar{\partial}\log(1-|z|^2)
\end{aligned}$$

where $\mathbb{B}^n = \{z \in \mathbb{C}^n; |z| < 1\}$. It is not hard to check that the curvature tensor of $\omega_{\mathbb{P}^n}$ satisfies

$$R_{i\bar{j}k\bar{\ell}} = g_{i\bar{j}}g_{k\bar{\ell}} + g_{i\bar{\ell}}g_{k\bar{j}},$$

and similarly for $\omega_{\mathbb{B}^n}$ with the opposite sign. Indeed, since $U(n+1,\mathbb{C})$ acts transitively on $\mathbb{P}^n$, it is enough to check the identity at the origin where the computation becomes elementary. In particular, we have

$$\mathrm{HSC}_{\omega_{\mathbb{P}^n}} \equiv 2, \quad \mathrm{Ric}(\omega_{\mathbb{P}^n}) = (n+1)\omega_{\mathbb{P}^n},$$

and similary with opposite sign for $\omega_{\mathbb{B}^n}$.

Remark 2.7. As soon as $n \geqslant 2$, the Kähler metrics $\omega_{\mathbb{P}^n}$ and $\omega_{\mathbb{B}^n}$ *do not* have constant sectional curvature in the usual Riemannian sense.

The following result can be found e.g. in [Tia00, Theorem 1.12].

Theorem 2.8. *(Metric uniformization) Let (X,ω) be a complete, simply connected Kähler manifold such that ω has constant holomorphic sectional curvature. Then up to scaling, (X,ω) is isometric to one of the three form spaces $(\mathbb{P}^n,\omega_{\mathbb{P}^n}), (\mathbb{C}^n,\omega_{\mathbb{C}^n}), (\mathbb{B}^n,\omega_{\mathbb{B}^n})$.*

3. Chern numbers and the uniformization theorem

3.1. **Chern classes and Chern numbers.** Let (X,ω) be a compact Kähler manifold. We have seen that there is an associated tensor

$$\Theta_\omega \in C^\infty(X, \mathrm{End}(T_X) \otimes \Omega^{1,1}_X)$$

which can be locally thought of as a $n \times n$ matrix whose entries are real $(1,1)$-forms. One can expand the expression

$$\det(I + t\Theta_\omega) = 1 + ta_1(\omega) + t^2 a_2(\omega) + \cdots$$

as a polynomial with coefficients in the algebra $C^\infty(X, \bigoplus_{p=0}^n \Omega^{p,p}_X)$.

We set $c_i(X,\omega) := a_i(\omega)$ for $i = 0,\ldots,n$. It is classical that $c_i(X,\omega)$ is a d-closed form of bigree (i,i) whose de Rham class is independent of the choice of ω. We define

$$c_i(X) := [c_i(X,\omega)] \in H^{2i}(X,\mathbb{R}).$$

Example 3.1. For $i = 1, 2$, we have the concise formulas

$$c_1(X, \omega) = \mathrm{tr}(\Theta_\omega) = \mathrm{Ric}(\omega)$$
$$c_2(X, \omega) = \frac{1}{2}\big((\mathrm{tr}(\Theta_\omega))^2 - \mathrm{tr}(\Theta_\omega \wedge \Theta_\omega)\big).$$

In particular, given any Kähler metric ω, the form $\mathrm{Ric}(\omega)$ represents $c_1(X)$.

Remark 3.2. It can be checked that the classes $c_i(X)$ are integral, i.e. live in the image of $H^{2i}(X, \mathbb{Z}) \to H^{2i}(X, \mathbb{R})$. In particular, they are constant under deformation of the complex structure. However, they are in general not invariant under diffeomorphism – this has to do with the fact that one uses $\mathbb{C}$-linear connections to construct the Chern classes. That is, it could be that the differentiable manifold X carries two different complex structure for which the Chern classes do not agrees. Or in other words, there could exist a C^∞ diffeomorphism $f : X \to Y$ between Kähler manifolds such that $f^* c_i(Y) \neq c_i(X)$. E.g changing the complex structure J into $-J$ yields a diffeomorphism of a given manifold but it has the effect to take $c_1(X)$ into $-c_1(X)$.

A Chern number is any numerical quantity (i.e. real number) obtained by intersecting Chern classes. For instance, the integers

$$c_1(X)^n = \int_X c_1(X, \omega)^n,$$
$$c_2(X) \cdot c_1(X)^{n-2} = \int_X c_2(X, \omega) \wedge c_1(X, \omega)^{n-2}$$
$$c_n(X) = \int_X c_n(X, \omega)$$

are Chern numbers.

Classically, the last quantity $c_n(X)$ coincides with the topological Euler characteristic

$$e(X) = \sum_{k=0}^{2n} (-1)^k h^k(X, \mathbb{R}).$$

Remark 3.3. The same construction actually applies to any complex (non-necessarily holomorphic) hermitian line bundle (E, h) to produce closed forms $c_i(E, h) \in C^\infty(X, \Omega_X^{i,i})$ and cohomology classes $c_i(E) \in H^{2i}(X, \mathbb{R})$.

3.2. **Kähler-Einstein metrics.**

Definition 3.4. Let X be a complex manifold. A Kähler metric ω is called Einstein (or Kähler-Einstein) if there exists $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}(\omega) = \lambda \omega.$$

Since $\mathrm{Ric}(t\omega) = \mathrm{Ric}(\omega)$ for any $t > 0$, one can always scale the KE metric ω so that the Einstein constant $\lambda \in \{-1, 0, 1\}$.

Remark 3.5. Carrying a Kähler-Einstein metric imposes strong constraints on the underlying complex manifold X, namely that the class $c_1(X)$ has a sign. More precisely, the Kähler-Einstein imposes $\lambda\omega \in c_1(X)$, so that $\pm c_1(X)$ contains either 0 or a Kähler metric.

For instance for $X = \mathbb{P}^1 \times \Sigma$ where Σ is a compact Riemann surface of genus $g \geqslant 2$, then $c_1(X) = \mathrm{pr}_{\mathbb{P}^1}^* c_1(\mathbb{P}^1) + \mathrm{pr}_\Sigma^* c_1(\Sigma)$ has no sign.

The following celebrated result of Yau [Yau78b] states that *any* form of the right type in the first Chern class of X can be represented by the Ricci curvature of some Kähler form. One of the most striking application is for manifolds X with $c_1(X) = 0$.

Theorem 3.6 (Yau's solution of the Calabi conjecture)**.** *Let (X, ω_0) be a compact Kähler manifold and let $\eta \in c_1(X)$ be a closed $(1,1)$-form. Then there exists a unique Kähler metric $\omega \in [\omega_0]$ such that*

$$\mathrm{Ric}(\omega) = \eta.$$

In particular, if $c_1(X) = 0$, each Kähler cohomology class carries a unique Ricci flat metric, i.e. a Kähler metric ω such that $\mathrm{Ric}(\omega) = 0$.

Yau's theorem allowed to construct the first example of a Ricci flat compact manifold which is not flat:

Corollary 3.7. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d = n+1$ for $n \geqslant 3$. Then X admits a Ricci flat, non-flat Kähler metric.*

Proof. By adjunction, we have

$$c_1(X) = (c_1(\mathbb{P}^n) - c_1(\mathcal{O}_{\mathbb{P}^n}(n+1)))|_X = 0.$$

Thanks to Theorem 3.6, it is enough to show that X does not carry a flat Kähler metric. Since a simply connected complete Riemannian manifold has to be isometric to the Euclidean space, the existence of a flat metric on X would force $\pi_1(X)$ to be infinite. But Lefschetz hyperplane section theorem shows that the natural map $\pi_1(X) \to \pi_1(\mathbb{P}^n) = \{1\}$ is an isomorphism. □

In the case where $c_1(X) < 0$, Aubin [Aub78] and Yau [Yau78b] show the existence and uniqueness of a Kähler-Einstein metric with negative Einstein constant.

Theorem 3.8 (Aubin-Yau theorem)**.** *Let X be a compact Kähler manifold such that $c_1(X) < 0$. Then there exists a unique Kähler metric such that*

$$\mathrm{Ric}(\omega) = -\omega.$$

Similarly to Corollary 3.7, we get:

Corollary 3.9. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > n+1$ for $n \geqslant 3$. Then X admits a Kähler-Einstein metric of negative scalar curvature which is not locally symmetric.*

Proof. By adjunction, we have

$$c_1(X) = (c_1(\mathbb{P}^n) - c_1(\mathcal{O}_{\mathbb{P}^n}(d)))|_X = c_1(\mathcal{O}_{\mathbb{P}^n}(n+1-d) < 0.$$

Thanks to Theorem 3.8, it is enough to show that X does not carry a locally symmetric Kähler metric. Since a simply connected complete, locally symmetric Kähler manifold with negative scalar curvature is non-compact, the existence of such a metric on X would force $\pi_1(X)$ to be infinite. But Lefschetz hyperplane section theorem shows again that the natural map $\pi_1(X) \to \pi_1(\mathbb{P}^n) = \{1\}$ is an isomorphism. □

3.3. **Uniformization by Chern numbers.** Let X be a compact Kähler manifold of dimension n. We have seen above a metric criterion guaranteeing that X is uniformized by $\mathbb{P}^n, \mathbb{C}^n$ or $\mathbb{B}^n$. The goal of this section is to provide a different criterion which is easier to check in practice.

Start with the baby case where T_X is holomorphically trivial, i.e. there exist n global holomorphic vector fields $\xi_1, \dots, \xi_n \in H^0(X, T_X)$ such that $(\xi_{1,x}, \dots, \xi_{n,x})$ generate $T_{X,x}$ for any $x \in X$. It is elementary to check that the Albanese torus $\mathrm{Alb}(X) = H^0(X, \Omega^1_X)^*/H_1(X, \mathbb{Z})$ has dimension n and that the Albanese map $a : X \to \mathrm{Alb}(X)$ is a local biholomorphism. In particular, a is finite étale and X is a torus itself.

If one only assumes that T_X is smoothly trivial (i.e. the trivializing vector fields ξ_i above are C^∞ and not necessarily holomorphic), one can find a flat hermitian metric on T_X (by declaring that the ξ_i form an orthonormal basis at any point) hence we have $c_i(X) = 0$ for all $i \geqslant 1$, cf Remark 3.3. This condition on the Chern classes certainly does not force X to be a torus, see Example 3.10 below exhibiting a torus quotient which is not itself a torus. We will later see (item (i) in Theorem 3.13) that having all (real) Chern classes vanish is equivalent to being a torus quotient.

Example 3.10. (A torus quotient need not be a torus) . Consider the elliptic curve $E = \mathbb{C}/\mathbb{Z}[\zeta]$ where ζ is a primitive root of unity order 3. One can easily check that the action of $\mathbb{Z}_3$ on E^3 defined by $\zeta \cdot (x, y, z) = (x + \frac{1}{3}, \zeta y, \zeta^2 z)$ is free, hence the quotient

$$X := E^3/\mathbb{Z}_3$$

is a compact Kähler manifold which is the target of the finite étale map $f : E^3 \to X$. Moreover, the canonical bundle of X is trivial. It is easy to check that $H^0(X, T_X) = H^0(E^3, T_{E^3})^{\mathbb{Z}_3}$ is generated by $\frac{\partial}{\partial x}$ hence X is not a torus. However, since f is étale, one has $0 = c_i(E^3) = f^* c_i(X)$ hence $c_i(X) = 0 \in H^{2i}(X, \mathbb{Q})$.

The starting point for the uniformization theorem is a computation that dates back to Chen-Ogiue [CO75].

Proposition 3.11. *Let (X, ω) be a compact Kähler-Einstein manifold of dimension $n \geqslant 2$ such that $\mathrm{Ric}(\omega) = \lambda\omega$ for some $\lambda \in \mathbb{R}$. Then ω has constant holomorphic sectional curvature if, and only if*

$$(2(n+1)c_2(X) - nc_1(X)^2) \cdot [\omega]^{n-2} = 0.$$

Remark 3.12. The proof will also show that assuming only that (X, ω) is compact KE, then $(2(n+1)c_2(X) - nc_1(X)^2) \cdot [\omega]^{n-2} \geqslant 0$.

Sketch of proof. Consider the traceless tensor

$$R^\circ =: R - \frac{\lambda}{n+1}(g_{i\bar{j}} g_{k\bar{\ell}} + g_{i\bar{\ell}} g_{k\bar{j}}),$$

see (2.6). One can prove, relying on the Einstein condition that

$$(2(n+1)c_2(X, \omega) - nc_1(X, \omega)^2) \wedge \omega^{n-2} = a_n |R^\circ|^2 \omega^n$$

where a_n is a positive dimensional constant. Integrating over X yields the result. □

As a corollary of Proposition 3.11, Theorems 3.6 and Theorem 3.8, we get

Theorem 3.13. *Let (X, ω) be a compact Kähler manifold.*

(1) If $c_1(X) = 0$, then $X = T/G$ is a torus quotient if and only if

$$c_2(X) \cdot [\omega]^{n-2} = 0.$$

(2) If $c_1(X) < 0$, then $X = \mathbb{B}^n/\Gamma$ is a ball quotient if and only if

$$(2(n+1)c_2(X) - nc_1(X)^2) \cdot c_1(X)^{n-2} = 0.$$

Proof. All there is left to proving is that a compact Kähler manifold (X, ω) whose universal cover $(\widetilde{X}, \widetilde{\omega})$ is isometric to the euclidean space $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ is a torus quotient. Since $\pi_1(X)$ acts isometrically and co-compactly on $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$, Bieberbach theorem shows that there exists a finite index subgroup $\Lambda \subset \pi_1(X)$ which is a lattice in $\mathbb{C}^n$ (more precisely it consists of translations by elements of a lattice). Since the natural map $\mathbb{C}^n/\Lambda \to X$ is finite étale, this concludes the proof. □

4. Positive curvature

This section aims to present the following two results "of topological flavor" whose proofs rely heavily on Yau's and Aubin-Yau's theorems, i.e. Theorem 3.6 and Theorem 3.8.

Theorem 4.1. *Let X be a compact Kähler manifold such that $c_1(X) > 0$. Then X is simply connected.*

Remark 4.2. Compact Kähler manifolds such that $c_1(X) > 0$ are often call Fano manifolds, or when $\dim X = 2$ del Pezzo surfaces.

Theorem 4.3. *Let X be a compact complex surface which is homotopy equivalent to $\mathbb{P}^2$. Then X is biholomorphic to $\mathbb{P}^2$.*

Remark 4.4. It is conjectured that a compact complex manifold X homeomorphic to $\mathbb{P}^n$ is biholomorphic to $\mathbb{P}^n$. The result is known if X is assumed to be Kähler, see e.g. the nice survey [Tos17].

4.1. Holomorphic Euler characteristic.

Definition 4.5. Let X be a compact complex manifold of dimension n. The holomorphic Euler characteristic of X is the integer

$$\chi(X) := \chi(X, \mathcal{O}_X) = \sum_{k=0}^{n} (-1)^k h^k(X, \mathcal{O}_X).$$

The fundamental theorem of Hirzebruch-Riemann-Roch gives a formula to compute $\chi(X)$ (and more generally $\chi(X, E)$ for any holomorphic vector bundle E on X) by integrating suitable Chern classes of E and X. In what follows, we will rely on the following important corollaries:

Theorem 4.6 (Noether formula)**.** *Let X be a compact complex surface. Then*

$$\chi(X) = \frac{1}{12}(c_2(X) + c_1(X)^2).$$

Moreover for any line bundle L on X we have

$$\chi(X, L) - \chi(X) = \frac{1}{2}(c_1(L)^2 + c_1(L) \cdot c_1(X)).$$

Theorem 4.7 (Multiplicativity of χ)**.** *Let $f : Y \to X$ be a finite étale cover of degree d between two compact complex manifolds. Then*

$$\chi(Y) = d \cdot \chi(X).$$

4.2. **Fano manifolds are simply connected.** This section is devoted to the proof of Theorem 4.1. An important observation is the following consequence of Kodaira vanishing theorem.

Lemma 4.8. *Let X be a compact Kähler manifold X such that $c_1(X) > 0$. Then $\chi(X) = 1$.*

Proof of Lemma 4.8. Recall that Kodaira vanishing theorem asserts that given a holomorphic line bundle $L \to X$ admitting a hermitian metric h such that $i\Theta(L,h)$ is a Kähler form, then

$$H^q(X, K_X \otimes L) = 0$$

for any $q > 0$. Here $K_X := \det(T^*_X)$. In particular we can apply the above result to $L := K_X^{-1}$ whose first Chern class $c_1(X)$ is Kähler by assumption. This yields $h^q(X, \mathcal{O}_X) = 0$ for any $q > 0$, and the lemma follows. □

We can now give the proof of Theorem 4.1.

Proof of Theorem 4.1. First, we claim that $\pi_1(X)$ is finite or, equivalently, that the universal cover $\widetilde{X}$ of X is compact. Indeed, fix a Kähler metric $\eta \in c_1(X)$. Thanks to Theorem 3.6, there exists another Kähler metric ω such that

$$\mathrm{Ric}(\omega) = \eta.$$

Since η is a Kähler metric, there exists for any $x \in X$ a constant $k_x > 0$ such that $\eta_x > k_x\omega_x$ (i.e. the difference is a Kähler metric). By compactness of X, one choose a positive constant $k > 0$ which is independent of x, i.e.

$$\mathrm{Ric}(\omega) \geqslant k\omega.$$

Denote by $\pi : \widetilde{X} \to X$ the universal cover of X. The induced Kähler metric $\widetilde{\omega} := \pi^*\omega$ on $\widetilde{X}$ still satisfies $\mathrm{Ric}(\widetilde{\omega}) \geqslant k\widetilde{\omega}$ hence Myers' theorem shows that $\widetilde{X}$ is compact, as claimed.

Since $\pi : \widetilde{X} \to X$ is finite, one can apply Theorem 4.7 to see that

$$\chi(\widetilde{X}) = \deg(\pi)\chi(X).$$

But since $c_1(\widetilde{X}) = \pi^*c_1(X)$ is positive, Lemma 4.8 yields $\chi(\widetilde{X}) = \chi(X) = 1$. Therefore $\deg(\pi) = 1$, i.e. π is an isomorphism and X is simply connected. □

4.3. **Topological characterization of $\mathbb{P}^2$.** Let us prove Theorem 4.3. We start by recalling that the cohomology algebra of $\mathbb{P}^2$ satisfies

$$H^\bullet(\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}[\alpha]$$

where $\alpha \in H^2(\mathbb{P}^2, \mathbb{Z})$ satisfies $\alpha^2 = 1$ and $c_1(\mathbb{P}^2) = 3\alpha$. Decomposing $\mathbb{P}^2 = \mathbb{C}^2 \sqcup \mathbb{P}^1 = \mathbb{C}^2 \sqcup \mathbb{C} \sqcup \{pt\}$, we also find $e(\mathbb{P}^2) = c_2(\mathbb{P}^2) = 3$.

Since X is homotopy equivalent to $\mathbb{P}^2$, we deduce

$$c_2(X) = e(X) = 3. \tag{4.1}$$

We proceed in three steps.

Step 1. X is a Kähler surface.

One could appeal to classification and use the result stating that a compact complex surface with an even first Betti number (in our case we have $b_1(X) = 0$) is Kähler.

We sketch below a more economical argument. One can write $H^2(X, \mathbb{Z}) = \mathbb{Z}\beta$ with $\beta^2 = 1$. Write

$$c_1(X) = \lambda\beta \tag{4.2}$$

for some $\lambda \in \mathbb{Z}$. Thanks to Theorem 4.6 and (4.1), we get

$$0 \leqslant \lambda^2 = c_1(X)^2 = 12\chi(X) - c_2(X) = 12\chi(X) - 3. \tag{4.3}$$

We infer that $\chi(X) \geqslant 1$, hence

$$c_1(X)^2 \geqslant 9. \tag{4.4}$$

Set $L := K_X$. Hirzerbruch-Riemann-Roch shows

$$h^0(X, L^{\otimes m}) + h^0(X, L^{\otimes(1-m)}) = \frac{(L^2)}{2}m^2 + o(m).$$

It is elementary to see that the image rational map $\phi : X \dashrightarrow \mathbb{P}^N$ induced by global sections of either $L^{\otimes m}$ or $L^{\otimes(-m)}$ for m large must have dimension two. One can resolve the indeterminacies of ϕ by blowing up points in X, yielding a bimeromophic map $X' \to X$ where X' is projective. Castelnuvo theorem shows that the successive blow downs of X' are projective, hence so is X.

Step 2. X is a del Pezzo surface.

Since X is Kähler, $H^2(X, \mathbb{R})$ is one-dimensional and $c_1(X) \neq 0$ by (4.4), we infer that one has either $c_1(X) < 0$ or $c_1(X) > 0$.

If $c_1(X) < 0$, the existence of a Kähler-Einstein metric (Theorem 3.8) yields

$$3c_2(X) - c_1(X)^2 \geqslant 0, \quad \text{i.e.} \quad c_1(X)^2 \leqslant 9.$$

with equality if and only if $X = \mathbb{B}^n/\Gamma$ is a ball quotient, see Proposition 3.11 and Remark 3.12. But (4.4) forces the equality to occur, yielding a contradiction with the simple connectedness of X.

Therefore one must have $c_1(X) > 0$, as announced in the title of the step.

Step 3. X isomorphic to $\mathbb{P}^2$.

Here, one can either rely on the classification of del Pezzo surfaces, whose list consists in $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the blow ups of $\mathbb{P}^2$ in at most 8 points. The only del Pezzo surface with $b_2 = 1$ is therefore $\mathbb{P}^2$.

Alternatively, one can invoke Kobayashi-Ochiai's theorem [KO73, Theorem 1.1] stating that a projective manifold X of dimension n endowed with a line bundle A satisfying $c_1(A)^n = 1$ and $h^0(X, A) \geqslant n + 1$ must be $\mathbb{P}^n$. To find such a line bundle A, we start by observing that the first Chern class map

$$c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$$

is an isomorphism. Indeed, the exponential exact sequence shows injectivity while the previous steps ensures that the image of c_1 is non-zero, hence of the form $d\mathbb{Z}\beta$ for some $d \geqslant 1$. The exponential exact sequence then yields an injection $\mathbb{Z}/d\mathbb{Z} \to H^2(X, \mathcal{O}_X)$ but since the latter is torsion-free (is it is a complex vector space) one must have $d = 1$. Denote by A the ample line bundle on X such that $c_1(A) = \pm\beta$. Clearly, $c_1(A)^2 = 1$ so it remains to see that $h^0(X, A) = 3$. By (4.2) and the positivity of $c_1(X)$, we have

$$-K_X = A^{\otimes 3}. \tag{4.5}$$

Thanks to Kodaira vanishing theorem, we get

$$h^q(X, A) = h^q(X, K_X \otimes A^{\otimes 4}) = 0, \quad q = 1, 2. \tag{4.6}$$

Combining (4.6) with Theorem 4.6, Lemma 4.8 and (4.5), we obtain

$$h^0(X, A) = \chi(X, A) = 1 + \frac{1}{2}(1 + 3) = 3,$$

as announced. This concludes the proof of the theorem.

5. Zero curvature

The goal of this section is to explain the following result.

Theorem 5.1. *Let X be a compact Kähler manifold such that $c_1(X) = 0 \in H^2(X, \mathbb{R})$. Then*

1. *K_X is torsion, i.e. there exists an integer $m \geqslant 1$ such that $K_X^{\otimes m} \sim \mathcal{O}_X$.*
2. *$\pi_1(X)$ is virtually abelian, i.e. it contains a (normal) subgroup of finite index which is abelian.*

Recall that the assumption $c_1(X) = 0 \in H^2(X, \mathbb{R})$ means that some tensor power $K_X^{\otimes m}$ ($m \in \mathbb{Z}^*$) is topologically trivial, i.e. admits a continuous (or smooth) section which does not vanish. This can be seen as a consequence of the following two facts:

1. A complex line bundle L is topologically trivial iff $c_1(L) = 0 \in H^2(X, \mathbb{Z})$. This follows from the exponential exact sequence along with the fact that the sheaves of smooth functions have partitions of unity, hence no higher cohomology.
2. The kernel of the natural map $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ is finite hence torsion (it coincides with $H^2(X, \mathbb{Z})_{\text{tor}}$).

Theorem 5.1 will be obtained as a consequence of the following result

Theorem 5.2. *Let X be a compact Kähler manifold such that $c_1(X) = 0 \in H^2(X, \mathbb{R})$. There exists a finite étale Galois cover $f : X' \to X$ such that*

$$X' = T \times Y$$

where $T = \mathbb{C}^k/\Lambda$ is a compact complex torus and Y is simply connected.

Remark 5.3. The manifold Y enjoys many geometric properties as described by the so-called Beauville-Bogomolov decomposition theorem, see [Bea83].

Theorem 5.2 implies Theorem 5.1. The Galois cover f yields an exact sequence

$$1 \to \pi_1(X') \simeq \mathbb{Z}^{2k} \to \pi_1(X) \to \mathrm{Gal}(f) \to 1$$

hence the second item of Theorem 5.1 follows.

Next, we claim that

$$K_Y \simeq \mathcal{O}_Y. \tag{5.1}$$

Indeed, we know that

$$0 = f^* c_1(X) = c_1(X') = c_1(Y) \in H^2(X, \mathbb{R}).$$

Since Y is simply connected and compact Kähler, we have $H^1(Y, \mathcal{O}_Y) = 0$. In particular, the integral first Chern class map

$$\mathrm{Pic}(Y) \to H^2(Y, \mathbb{Z})$$

is an injection. Since $H^2(X, \mathbb{Z})_{\mathrm{tor}} \simeq H_1(X, \mathbb{Z})_{\mathrm{tor}} = 0$ thanks to the universal coefficients theorem, the natural map $H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{R})$ is injective as well, hence the identity $c_1(K_Y) = 0 \in H^2(Y, \mathbb{R})$ implies $K_Y \simeq \mathcal{O}_Y$.

The Galois group G of f naturally acts on K_Y. So if we choose a trivialization $\sigma \in H^0(Y, K_Y)$, then the section

$$s := \bigotimes_{g \in G} g \cdot \sigma \in H^0(Y, K_Y^{\otimes |G|})$$

is a G-invariant trivialization of $K_Y^{\otimes |G|} \simeq f^* K_X^{\otimes |G|}$ hence it descends to a trivialization of $K_X^{\otimes |G|}$, as desired. □

Proof of Theorem 5.2. Thanks to Yau's theorem (Theorem 3.6), X admits a Ricci flat Kähler metric ω. Let $\pi : \widetilde{X} \to X$ be the universal cover of X and let $\widetilde{\omega} := \pi^* \omega$. The Kähler manifold $(\widetilde{X}, \widetilde{\omega})$ is simply connected, complete and Ricci flat. Hence Cheeger-Gromoll splitting theorem [CG71, Theorem 3] allows one to find a holomorphic, isometric splitting

$$(\widetilde{X}, \widetilde{\omega}) = (\mathbb{C}^k, \omega_{\mathbb{C}^k}) \times (Y, \omega_Y)$$

where Y is *compact*. Moreover, such a splitting is unique up to products of isometries in each factor.

Uniqueness implies that any element $u = (u_1, u_2) \in \pi_1(X)$ viewed as a holomorphic isometry of $(\widetilde{X}, \widetilde{\omega})$ is such that $u = (u_1(z), u_2(y))$ if one writes $x = (z, y)$ with $z \in \mathbb{C}^k$ and $y \in Y$. In particular, there is a natural splitting $\pi_1(X) = G \times H$ with $G \subset \mathrm{Isom}(\mathbb{C}^k, \omega_{\mathbb{C}^k})$ and $H \subset (Y, \omega_Y)$ acting freely and properly. Since Y is compact, H must be finite and $\mathbb{C}^k/G$ is compact. Bieberbach's theorem ensures that the subgroup of translations in G has finite index. In particular there exists a lattice $\Lambda \subset \mathbb{C}^k$ which can be realized as a finite index normal subgroup of G. Set $T := \mathbb{C}^n/\Lambda$ and $X' := T \times Y$. The finite group $G/\Lambda \times H$ acts freely on X' and the quotient $X'/(G/\Lambda \times H)$ identifies naturally with X. Theorem 5.2 follows. □

6. Negative curvature

The main result that we will discuss is a beautiful theorem due to Nadel [Nad90]. Before giving the statement, let us introduce some notations and context.

Let X be a compact complex manifold such that $c_1(X) < 0$. In particular, X is automatically projective algebraic. We have the following result:

Proposition 6.1. *The group of automorphism* $\mathrm{Aut}(X)$ *of a projective manifolds with* $c_1(X) < 0$ *is finite.*

Proof. The starting point is Kodaira-Akizuki-Nakano vanishing which yields

$$H^0(X, T_X) \simeq H^n(X, K_X \otimes \Omega_X) = H^{1,n}(K_X) = 0 \tag{6.1}$$

since $1 + n > n$ and K_X is positive.

From there, there are several possible routes. First, one can observe that any automorphism $f \in \mathrm{Aut}(X)$ induces an isomorphism $df : T_X \to f^*T_X$ and therefore induces an isomorphism $f^*K_X \simeq K_X$. In particular f induces an action on $V_m := H^0(X, K_X^{\otimes m})$ for any $m \geqslant 1$. Take m large enough so that the induced map $\phi_{mK_X} : X \to \mathbb{P}(V_m^*)$ is an embedding. We get an injective group morphism

$$\mathrm{Aut}(X) \to \mathrm{PGL}(V_m^*)$$

realizing $\mathrm{Aut}(X)$ as a closed algebraic subgroup of $\mathrm{PGL}(V_m^*)$. In particular, it is affine with finitely many connected components. But $\mathrm{Aut}^\circ(X) = \{1\}$ by (6.1), hence the lemma.

Alternatively, use the uniqueness of the Kähler-Einstein metric ω provided by Theorem 3.8 to see that we have an injection

$$\mathrm{Aut}(X) \to \mathrm{Isom}(X, \omega).$$

But the latter is a compact Lie group by Myers-Steenrod theorem, and the conclusions follows from (6.1). □

Now what about the universal cover $p : \widetilde{X} \to X$?

There are examples where X is simply connected (hypersurfaces in the projective space of large enough degree, see Corollary 3.9), and some where $\pi_1(X)$ is infinite. For instance $\widetilde{X}$ can be a symmetric domain (e.g. if X is a ball quotient) in which case $\widetilde{X}$ has a large group of automorphisms. There exist also surfaces X (called Kodaira surfaces) which are fibrations over a hyperbolic curve and whose universal cover is a domain in $\mathbb{C}^2$ whose automorphism group is discrete.

Thanks to Aubin-Yau's theorem (Theorem 3.8), X admits a (unique) Kähler-Einstein metric ω, say with Einstein constant -1. Denote by g the associated Riemannian metric on X and set $\widetilde{\omega} := p^*\omega$ (resp. $\widetilde{g} := p^*g$). The manifold $(\widetilde{X}, \widetilde{\omega} := p^*\omega)$ is complete, Kähler-Einstein. Moreover, if follows from Yau's maximum principle [Yau78a] that $\widetilde{\omega}$ is the *unique* complete Kähler-Einstein metric (with Einstein constant -1). In particular, any element $f \in \mathrm{Aut}(\widetilde{X})$ in the group of biholomorphic transformations of X preserves $\widetilde{\omega}$, i.e. $f^*\widetilde{\omega} = \widetilde{\omega}$ (and $f^*\widetilde{g} = \widetilde{g}$, too). This allows to view

$$\mathrm{Aut}(\widetilde{X}) \subset \mathrm{Isom}(\widetilde{X}, \widetilde{g})$$

as a closed subgroup of the group of isometries of $(\widetilde{X}, \widetilde{g})$. Thanks to a theorem of Myers and Steenrod, the latter group is a real Lie group (of finitely many dimensions but possibly with infinitely many connected components if $\widetilde{X}$ is not compact) hence $\mathrm{Aut}(\widetilde{X})$ is a real Lie group as well, cf Section 6.1 for a description of the associated Lie algebras.

Nadel's theorem describes the structure of the connected component $\mathrm{Aut}^\circ(\widetilde{X})$ of the identity in $\mathrm{Aut}(\widetilde{X})$.

Theorem 6.2 ([Nad90]). *Let X be a compact Kähler manifold such that $c_1(X) < 0$. Then the connected Lie group $\mathrm{Aut}^\circ(\widetilde{X})$ is semisimple, without compact factor.*

Of course, this theorem is only interesting when $\pi_1(X)$ is infinite (see Proposition 6.1).

In the next two sections, we will recall some basic facts on holomorphic vector fields and Lie algebras (e.g. the notions of semisimplicity and compact factor). Then we will digress on the notion of slope stability for vector bundles and give an application of Aubin-Yau's theorem in that context. In the last section, we will give the proof of Theorem 6.2.

6.1. **Infinitesimal isometries of Kähler manifolds.** Let (X, ω) be a complete Kähler manifold and consider the group $\mathrm{Isom}(X, \omega)$ of *biholomorphic* maps $f : X \to X$ such that $f^*\omega = \omega$. This group embeds in the group $\mathrm{Isom}(X, g)$ of isometries of the complete Riemannian manifold (X, g) where g is the Riemannian metric associated to ω, i.e. $g(u, v) = \omega(u, Jv)$ for $u, v \in T_X$. Myers-Steenrod theorem ensures that $\mathrm{Isom}(X, g)$ is a real Lie group (of finite dimension, but possibly infinitely many connected components). Its Lie algebra can be identified with

$$\{u \in \mathcal{C}^\infty(X, T_X); \mathcal{L}_u g = 0\}.$$

Here, T_X is though of as the real tangent bundle $T_{X,\mathbb{R}}$. This identification relies on the fact that a Killing vector field u (i.e. $\mathcal{L}_u g = 0$) is complete in the sense that it induces a globally defined flow $(\phi_t)_{t\in\mathbb{R}}$ of diffeomorphisms of X.

Recall that a vector field $u \in C^\infty(X, T_X)$ is holomorphic iff $\bar\partial u = 0$ iff $\mathcal{L}_u J = 0$ (i.e. $[u, Jv] = J[u, v]$ for any local vector field v) iff the local flow of u consists of holomorphic diffeomorphisms. Since

$$\mathcal{L}_u g(\cdot, \cdot) = (\mathcal{L}_u \omega)(\cdot, J\cdot) + \omega(\cdot, (\mathcal{L}_u J)\cdot),$$

a holomorphic vector field is Killing for g iff it is symplectic, i.e. $\mathcal{L}_u \omega = 0$. This observation allows to identify the (real) Lie algebra of $\mathrm{Isom}(X, \omega)$ to

$$\mathfrak{isom}(X, \omega) := \{u \in H^0(X, T_X);\ \mathcal{L}_u \omega = 0\}.$$

If one prefers to think in terms of the complexified tangent bundle via the isomorphism (2.3), one can view the former Lie algebra as

$$\{\xi \in C^\infty(X, T^{1,0}_X);\ \nabla^{0,1}\xi = 0 \text{ and } \mathcal{L}_{\mathrm{Re}(\xi)}\omega = 0\}.$$

Here, $\nabla^{0,1}$ is the $(0, 1)$-part of the connection on $T_{X,\mathbb{C}}$ induced by the Chern connection of (T_X, ω).

Remark 6.3. The real Lie algebra $\mathfrak{isom}(X,\omega)$ need not be complex subalgebra of the complex Lie algebra $H^0(X,T_X)$, see e.g. [Kob95, III.1 Theorem 1.3]. E.g. if $0 \neq u, Ju \in H^0(X,T_X)$ are Killing, then u is parallel so that (X,ω) has a flat factor in its local de Rham decomposition, which prevents $\mathrm{Ric}\,\omega$ from being definite. Also, since $[u,Ju] = J[u,u] = 0$, (u,Ju) generates a one-dimensional complex Lie group of automorphisms of X, hence a holomorphic action of $\mathbb{C}$ on X. In particular, X cannot be complex hyperbolic.

6.2. **Basics on Lie algebras.** Recall that a Lie algebra $\mathfrak{g}$ is simple if it is non-abelian and any ideal $\mathfrak{h}$ (i.e. $[\mathfrak{h},\mathfrak{g}] \subset \mathfrak{h}$) is either zero or the whole $\mathfrak{g}$. By definition, a Lie algebra is semisimple if it can be decomposed

$$\mathfrak{g} = \bigoplus_{i\in I} \mathfrak{g}_i \tag{6.2}$$

as a direct sum of simple Lie algebras $\mathfrak{g}_i$. In particular, the Lie subalgebras $\mathfrak{g}_i \subset \mathfrak{g}$ are ideals.

In particular, we have

$$\{\text{simple ideals of } \mathfrak{g}\} = \{0\} \cup \{\mathfrak{g}_i, i \in I\} \tag{6.3}$$

Indeed, let $0 \neq \mathfrak{h} \subset \mathfrak{g}$ be a simple ideal. Pick $i \in I$. Then $[\mathfrak{h},g_i]$ is an ideal in both $\mathfrak{g}_i$ and $\mathfrak{h}$. Therefore, it is either zero or we have $[\mathfrak{h},g_i] = \mathfrak{g}_i = \mathfrak{h}$. If $[\mathfrak{h},g_i] = 0$, then the image of the natural projection $\mathfrak{h} \to \mathfrak{g}_i$ lies in the center $Z(\mathfrak{g}_i)$ of $\mathfrak{g}_i$, but the latter is zero by the definition of a simple Lie algebra. Therefore $\mathfrak{h} \subset \oplus_{j\neq i}\mathfrak{g}_j$ and one can iterate.

Recall the definition of the Cartan-Killing form

$$B_{\mathfrak{g}}(x,y) := \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y), \quad x,y \in \mathfrak{g}.$$

The form $B_{\mathfrak{g}}$ is invariant under any automorphism of $\mathfrak{g}$. Moreover, a theorem of Cartan states that $\mathfrak{g}$ is semisimple if and only if $B_{\mathfrak{g}}$ is definite (i.e. $B_{\mathfrak{g}}(x,\mathfrak{g}) = 0$ implies $x = 0$). From the decomposition (6.2), one infers that $(B_{\mathfrak{g}})|_{\mathfrak{g}_i\times\mathfrak{g}_i} = B_{\mathfrak{g}_i}$.

We say that $\mathfrak{g}_i$ is a compact factor if $B_{\mathfrak{g}}$ is negative definite in restriction to $\mathfrak{g}_i$, i.e. if $B_{\mathfrak{g}_i}$ is negative definite (write $B_{\mathfrak{g}_i} < 0$ for short). Set $I_- := \{i \in I, B_{\mathfrak{g}_i} < 0\}$ and

$$\mathfrak{g}_- := \bigoplus_{j\in I_-} \mathfrak{g}_j.$$

By (6.3) and invariance of $B_{\mathfrak{g}}$, an automorphism of $\mathfrak{g}$ permutes the $\mathfrak{g}_i$ and fixes $\mathfrak{g}_-$.

Let $\mathfrak{g}$ be a Lie algebra. One says that $\mathfrak{g}$ is solvable if the derived series

$$\mathfrak{g}^{(0)} = 0, \quad \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)},\mathfrak{g}^{(n)}]$$

satisfies $\mathfrak{g}^{(n)} = 0$ for n large. One can consider a solvable ideal of $\mathfrak{g}$ of maximal dimension. Since the sum of two solvable ideals in $\mathfrak{g}$ is still solvable, such an ideal is unique. It is called the radical of $\mathfrak{g}$ and it is denoted by $\mathrm{rad}(\mathfrak{g})$. Moreover, one has

$$\mathfrak{g} \text{ is semisimple} \iff \mathrm{rad}(\mathfrak{g}) = 0.$$

Assume that $\mathrm{rad}(\mathfrak{g}) \neq 0$. Then the last non-zero element of the series $(\mathrm{rad}(\mathfrak{g})^{(n)})_{n\in\mathbb{N}}$ is a non-zero *abelian* ideal. Call it $\mathrm{ab}(\mathfrak{g})$ (this terminology is highly non-standard).

6.3. **Slope semistability and Hermite-Einstein metrics.** Let (X,ω) be a compact Kähler manifold of dimension n and let E be a holomorphic vector bundle of rank r on X. Set $\alpha := [\omega] \in H^2(X,\mathbb{R})$. If $F \subset E$ is a subsheaf (here we identify the vector bundle E with its locally free sheaf of sections $\mathcal{O}_X(E)$), we define the slope of F with respect to α by

$$\mu_\alpha(F) = \frac{1}{\mathrm{rk}(F)}\, c_1(F)\cdot\alpha^{n-1}.$$

Recall that F need not be locally free so that $c_1(F)$ is defined by $c_1(\det F)$, where $\det F := (\Lambda^r F)^{**}$ is a line bundle.

Given an exact sequence of sheaves

$$0 \to E_1 \to E \to E_2 \to 0$$

it is elementary to check that

$$\mu_\alpha(E) = \frac{r_1}{r}\mu_\alpha(E_1) + \frac{r_2}{r}\mu_\alpha(E_2) \tag{6.4}$$

is a convex combination of slopes of E_1 and E_2. Here, $r_i := \mathrm{rk}(E_i)$ and $r = r_1 + r_2 = \mathrm{rk}(E)$.

Definition 6.4. One says that E is α-semistable if for any subsheaf $0 \neq F \subset E$, one has

$$\mu_\alpha(F) \leqslant \mu_\alpha(E).$$

Equivalently, one has $\mu_\alpha(Q) \geqslant \mu_\alpha(E)$ for any non-zero quotient $E \to Q \to 0$.

Definition 6.5. A hermitian metric h on E is called Hermite-Einstein (relatively to the Kähler metric ω) if there is $\lambda \in \mathbb{R}$ such that

$$\frac{i}{2\pi}\Theta(E,h)\wedge\omega^{n-1} = \lambda \mathrm{Id}_E \otimes \omega^n.$$

Note that the constant $\lambda = \lambda_E$ is a priori determined by E via the formula $c_1(E)\cdot\alpha^{n-1} = \lambda_E r\alpha^n$.

The following result is due to Kobayashi-Lübke:

Theorem 6.6 (Kobayashi-Lübke inequality)**.** *If E admits a Hermite-Einstein metric with respect to ω, then E is $[\omega]$-semistable.*

Proof. We will show that when $F \subset E$ is a subbundle, then $\mu_\alpha(F) \leqslant \mu_\alpha(E)$. The case of non-locally free subsheaves is treated in [Kob87, V.8]. Let h_F be the hermitian metric induced by h on E. A classical formula due to Griffiths shows that

$$i\Theta(F,h_F) \leqslant \mathrm{pr}_F\, i\Theta(E,h)|_F$$

where $\mathrm{pr}_F : E \to F$ is the orthogonal projection with respect to h and the inequality has to be understood in the sense of Griffiths. In particular, wedging with ω^{n-1} and taking the endomorphism trace preserves the inequality

so that

$$\begin{aligned}\mathrm{tr}_{\mathrm{End}}(i\Theta(F,h_F))\wedge\omega^{n-1} &\leqslant \mathrm{tr}_{\mathrm{End}}\left(\mathrm{pr}_F i\Theta(E,h)|_F\wedge\omega^{n-1}\right)\\ &= \lambda_E \mathrm{tr}_{\mathrm{End}}\left(\mathrm{pr}_F(\mathrm{Id}_E|_F)\right)\omega^n\\ &= \lambda_E \mathrm{rk}(F)\,\omega^n.\end{aligned}$$

Multiplying both sides by $\frac{1}{2\pi\cdot\mathrm{rk}(F)}$ and integrating over X then yields

$$\mu_\alpha(F)\leqslant\lambda_E\alpha^n=\mu_\alpha(E),$$

which concludes the proof. □

As a Corollary of Theorem 3.8 and Theorem 6.6, we deduce:

Corollary 6.7. *Let X be a compact Kähler manifold such that $c_1(X)<0$. Then T_X is K_X-semistable.*

Proof. Let ω be the Kähler-Einstein metric on X provided by Theorem 3.8. In normal coordinates at a fixed point (see (2.1)), we have

$$\omega^{n-1}=(n-1)!\sum_k \widehat{id z_k\wedge d\bar{z}_k},$$

as well as $R^{\ j}_{i\ k\bar{\ell}}=R_{i\bar{j}k\bar{\ell}}$ and $\sum_k R_{i\bar{j}k\bar{k}}=-\delta_{ij}$ so that

$$\begin{aligned}n\Theta_\omega\wedge\omega^{n-1} &= R_{i\bar{j}k\bar{l}}\partial_i^*\partial_j\otimes\omega^n\\ &= -\mathrm{Id}_{T_X}\otimes\omega^n.\end{aligned}$$

In other words, the hermitian metric induced on T_X by ω is Hermite-Einstein. The corollary follows. □

6.4. **Holomorphic vector fields on $\widetilde{X}$.** In this section, we work under the assumptions of Theorem 6.2. That is, X is a compact Kähler manifold with $c_1(X)<0$, we denote by $p:\widetilde{X}\to X$ the universal cover and we set $\Gamma:=\mathrm{Aut}(p)\simeq\pi_1(X)$.

Proposition 6.8. *Let $0\neq V\subset H^0(\widetilde{X},T_{\widetilde{X}})$ be a finite-dimensional, Γ-invariant complex vector subspace. Let $\mathcal{M}(\widetilde{X})$ be the field of meromorphic functions of $\widetilde{X}$. Then*

1. *The natural map $V\otimes_{\mathbb{C}}\mathcal{M}(\widetilde{X})\to\mathcal{M}(\widetilde{X},T_{\widetilde{X}})$ is not injective.*
2. *There is no Γ-invariant hermitian product on V.*

Remark 6.9. Let us make a couple of remarks on the first item above.

(1) One can rephrase the conclusion by saying that if $v_1,\dots,v_r$ is a $\mathbb{C}$-basis of V, then there are meromorphic functions (not all zero) $f_i\in\mathcal{M}(X)$ such that $\sum f_i v_i=0\in\mathcal{M}(\widetilde{X},T_{\widetilde{X}})$. Of course one has $\dim_{\mathcal{M}(X)}V\otimes_{\mathbb{C}}\mathcal{M}(\widetilde{X})=\dim_{\mathbb{C}}V$.
(2) As a consequence, one cannot find such a V of (complex) dimension one. That is, there is no non-zero holomorphic vector field ξ on $\widetilde{X}$ such that Γ preserves $\mathbb{C}\,\xi$.

Proof. Out of V, one can cook up a trivial vector bundle $\widetilde{E} := V \otimes \mathcal{O}_{\widetilde{X}}$ endowed with an action of Γ. Moreover, one has an inclusion $\widetilde{E} \subset T_{\widetilde{X}} = p^*T_X$ of vector bundles. The vector bundle $\widetilde{E}$ can be descended (although non-necessarily trivially) to X by setting

$$E := (\tilde{X} \times V)/\Gamma.$$

That is, E is a flat vector bundle on X such that $p^*E \simeq \widetilde{E}$. In particular, all Chern classes $c_i(E)$ vanish (for $i \geqslant 1$). Moreover, the inclusion $\widetilde{E} \subset T_{\widetilde{X}}$ yields a non-zero map

$$f : E \to T_X$$

of coherent sheaves.

Set $\alpha = -c_1(X)$. By Corollary 6.7, we have

$$\mu_\alpha(\mathrm{im}(f)) \leqslant \mu_\alpha(T_X) < 0. \tag{6.5}$$

Since $0 = \mu_\alpha(E)$ is a convex combination of $\mu_\alpha(\ker(f))$ and $\mu_\alpha(\mathrm{im}(f))$ (see (6.4)), we infer that

$$\ker(f) \neq 0.$$

In particular, there exists a non-zero meromorphic section $s \in \mathcal{M}(X, \ker(f))$. Pick a basis $v_1, \ldots, v_r$ of the vector space V (note that the v_i are not Γ-invariant). The pullback p^*s can be identified with an r-tuple of meromorphic functions $(f_1, \ldots, f_r)$ on $\widetilde{X}$ such that

$$\sum_{i=1}^{r} f_i v_i = 0 \quad \text{on} \quad \widetilde{X}.$$

This shows the first item.

As for the second item, argue by contradiction and choose a Γ-invariant hermitian metric h on V. It induces a *flat* hermitian metric on the vector bundle $\widetilde{E}$. In particular, it would force $\widetilde{E}$ to be α-semistable thanks to Theorem 6.6. But then one would have

$$0 = \mu_\alpha(\widetilde{E}) \leqslant \mu_\alpha(\mathrm{im}(f)),$$

in contradiction with (6.5). The proposition follows. □

As a consequence of item (i) in Proposition 6.8, one can show the following.

Lemma 6.10. *Let $V \subset \{u \in H^0(X, T_X);\ \mathcal{L}_u\omega = 0\}$ be a real Γ-invariant subspace. Then $[V, V] \neq 0$.*

Proof. Argue by contradiction. Then Proposition 6.8 applied to the Γ-invariant subspace $V \otimes_{\mathbb{R}} \mathbb{C} \subset H^0(X, T_X)$ allows us to find element $v_1, \ldots, v_s, v_{s+1} \in V$ with $s \geqslant 1$ such that

(1) $(v_1, \ldots, v_s, v_{s+1})$ is free over $\mathbb{R}$.
(2) The vector fields $v_1, \ldots, v_s$ are free over $\mathcal{M}(\widetilde{X})$.
(3) $v_{s+1} = \sum_{j=1}^{s} f_j v_j$ for some $f_j \in \mathcal{M}(\widetilde{X})$.
(4) $[v_j, v_k] = 0$, for all $1 \leqslant j, k \leqslant s+1$.

Note that

$$(\mathrm{Im}(f_1),\dots,\mathrm{Im}(f_s)) \neq (0,\dots,0). \tag{6.6}$$

Indeed, the f_j's would otherwise be constant functions, so that (3) would contradict (1). Moreover, taking the Lie bracket of (3) with v_k and using successively (4) and (2), we get

$$\forall j,k=1,\dots,s, \quad v_k\cdot f_j = 0. \tag{6.7}$$

Thanks to (2) and (4), one can work in suitable holomorphic charts near some point $\widetilde{x}\in\widetilde{X}$ such that

$$\forall j=1,\dots,s, \quad v_j=\partial_{z_j} \quad \text{and} \quad f_j \text{ is holomorphic.}$$

In the following, we use the notations

$$z_j = x_j+iy_j, \quad \text{and} \quad dV := i^n dz_1\wedge d\bar{z}_1\wedge\ldots\wedge dz_n\wedge d\bar{z}_n.$$

Since f_j is holomorphic, we deduce from (6.7)

$$\forall j,k=1,\dots,s, \quad \partial_{x_k} f_j = \partial_{y_k} f_j = 0. \tag{6.8}$$

Define a smooth positive function h by

$$\widetilde{\omega}^n = h dV.$$

By assumption, we have

$$\mathcal{L}_{\mathrm{Re}(v)}\widetilde{\omega}^n = 0 \tag{6.9}$$

as well. Since $\mathcal{L}_{\partial_{x_j}} dV = \mathcal{L}_{\partial_{y_j}} dV = 0$, we infer

$$\forall j=1,\dots,s, \quad \partial_{x_j} h = 0. \tag{6.10}$$

Define $a_j := \mathrm{Re}(f_j)$, $b_j := \mathrm{Im}(f_j)$ and consider the vector field

$$\mathrm{Re}(v_{s+1}) = \frac{1}{2}\sum_{j=1}^{s}(a_j\partial_{x_j}+b_j\partial_{y_j}).$$

We claim that

$$\sum_{j=1}^{s} b_j\,\partial_{y_j} h = 0. \tag{6.11}$$

Indeed, using successively (6.9), (6.10) and (6.8), we get

$$\begin{aligned}
0 &= \mathcal{L}_{\mathrm{Re}(v_{s+1})}\widetilde{\omega}^n \\
&= (\mathrm{Re}(v_{s+1})\cdot h)dV + h\sum_{j=1}^{s}\left(da_j\wedge\iota_{\partial_{x_j}}(dV) + db_j\wedge\iota_{\partial_{y_j}}(dV)\right) \\
&= \Big(\sum_{j=1}^{s} b_j\,\partial_{y_j} h\Big)dV,
\end{aligned}$$

and (6.11) follows. Applying ∂_{y_k} to (6.11) and using (6.8), one gets

$$\forall k=1,\dots,s, \quad \sum_{j=1}^{s} b_j\,\partial^2_{y_j y_k} h = 0. \tag{6.12}$$

Define a vector field of type $(1,0)$ by

$$\xi := \sum_{j=1}^{s} b_j \partial_{z_j},$$

which is not zero thanks to (6.6). We have

$$\begin{aligned} |\xi|^2_{\widetilde{\omega}} &= -\operatorname{Ric}(\widetilde{\omega})(\xi, \bar{\xi}) \\ &= \frac{1}{4}\left(\frac{1}{h} \sum_{j,k=1}^{s} b_k b_j \partial^2_{y_j y_k} h - \frac{1}{h^2} \Big(\sum_{j=1}^{s} b_j \partial_{y_j} h \Big)^2 \right) \end{aligned}$$

and the latter quantity vanishing thanks to (6.11) and (6.12). Therefore we have reached the desired contradiction. □

We can now finish the proof of Nadel's theorem.

Proof of Theorem 6.2. Denote by

$$\mathfrak{g} := \{ u \in H^0(X, T_X);\ \mathcal{L}_u \omega = 0 \}$$

the (real) Lie algebra of $\operatorname{Aut}(\widetilde{X})$. Lemma 6.10 shows that there exists no non-zero abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which is Γ-invariant. In particular, this applies to the algebra $\operatorname{ab}(\mathfrak{g})$ which is indeed abelian and Γ-invariant. This shows that $\operatorname{ab}(\mathfrak{g}) = 0$, hence $\operatorname{rad}(\mathfrak{g}) = 0$, too, and $\mathfrak{g}$ is semisimple.

If $\mathfrak{g}$ had a non-zero compact factor $\mathfrak{g}_-$, the latter would induce a Γ-invariant subspace of $\mathfrak{g}$ which is negative definite for the Killing form (the latter being Γ-invariant as well). Consider the complexified Lie algebra $\mathfrak{g}_- \otimes \mathbb{C}$, extend $B_{\mathfrak{g}_-}$ $\mathbb{C}$-bilinearily and set $H(x,y) = B^{\mathbb{C}}_{\mathfrak{g}_-}(x, \bar{y})$ for any $x, y \in \mathfrak{g}_- \otimes \mathbb{C}$. It is straighforward to check that H is hermitian negative definite and still Γ-invariant. This contradicts item (ii) of Proposition 6.8. Theorem 6.2 follows. □

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