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GEOMETRIC APPLICATIONS OF SINGULAR KÄHLER-EINSTEIN METRICS

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Highlighted publications

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[BGL22] B. Bakker, H. Guenancia & C. Lehn – "Algebraic approximation and the decomposition theorem for Kähler Calabi-Yau varieties", *Invent. Math.* **228** (2022), no. 3, p. 1255–1308.

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INTRODUCTION

Given a compact Kähler manifold X, a Kähler-Einstein metric is a Kähler metric ω such that its Ricci form is proportional to the Kähler form, i.e.

$$\operatorname{Ric}\omega = \lambda\omega$$

for some $\lambda \in \mathbb{R}$. One can always rescale ω so that $\lambda \in \{-1, 0, 1\}$. The existence of such a metric ω strongly constrains the geometry of X. The first constraint is cohomological, since $c_1(X)$ must have a sign (that of λ).

The celebrated theorems of Aubin-Yau and Yau (1978) show that when $\lambda \in \{-1, 0\}$ is non-positive, the necessary cohomological condition for the existence of a Kähler-Einstein metric is also a sufficient condition. It is hard to overestimate the importance of that result in complex geometry, as it not only had a tremendous number of applications, but it also opened the door to new fields of research.

The theorems were later extended to some non-compact manifolds (e.g. quasi-projective manifolds) in the late 80s and early 90s, and to singular compact Kähler spaces another twenty years later, motivated by the spectacular progress made in the Minimal Model Program. Due to the pluripotential theoretic nature of their construction, the geometry of these so-called singular Kähler-Einstein metrics is quite mysterious near the singularities of the space, making it quite difficult to derive geometric applications as it had been done for compact Kähler-Einstein manifolds.

The present memoir is mainly devoted to explaining a number of meaningful geometric applications of the existence of singular Kähler-Einstein metrics, although it will also survey a couple of results about families and degenerations of such metrics. We have decided to organize the results by the sign of the curvature, even though some techniques and ideas are common throughout the text.

The first part deals with negative curvature. We extend stability results for the tangent bundle of a Kähler-Einstein manifold to singular settings, which then allows us to derive a most general Miyaoka-Yau inequality, valid on any projective minimal model. Surprinsingly, some of the techniques involved in the proof of these results help answer a question of Lang on complex hyperbolicity in the particular setting of compact Kähler manifolds with negative holomorphic sectional curvature. The next paragraph is devoted to proving the psh variation of the relative Kähler-Einstein metric of a family of manifolds of general type, generalizing the celebrated case of families of canonically polarized manifolds. Finally, we give two examples of situations where one can degenerate a negative Kähler-Einstein metric with a cone angle along a smooth divisor by letting the cone angle go to zero. At the limit, one recovers classical complete Kähler-Einstein metrics: one has Poincaré singularities and the other is the complex hyperbolic cusp. In passing, this shows that one can close the cusp in the complex hyperbolic setting, up to working with orbifolds.

The second part focuses on the zero curvature case. The highlight of this section is probably the singular analogue of the Beauville-Bogomolov decomposition theorem, for compact Kähler spaces with klt singularities and zero first Chern class. Its proof relies on the result in the projective algebraic case by deformation theoretic arguments and singular Ricci flat Kähler metrics play a decisive role. In the first paragraph, we explain three intertwined intermediate results about such varieties: the Bochner principle (reflexive holomorphic tensors are parallel with respect to the singular KE metric), the étale triviality of the Albanese map and the computation of the holonomy of the KE metric on the regular locus. We also explain how these three results contributed to the proof of the decomposition theorem in the algebraic and transcendental setting. Capitalizing on the decomposition theorem, we provide a numerical criterion involving the vanishing of the first two Chern classes to characterize possibly singular quotients of complex tori. Finally, we switch gears and study families of log Calabi-Yau manifolds. First, we disprove a folkore conjecture asserting that the relative Ricci flat Kähler metric should be semipositive. Next, we prove a criterion of local triviality for the family involving the hermitian flatness of direct image of the log pluricanonical bundle. Two geometric applications are derived; the first one shows that the Albanese map of a log Calabi-Yau manifold is locally trivial while the second establishes the bigness of the direct image of the log pluricanonical bundle of an effective family of log Calabi-Yau manifolds.

The third part contains two results involving Fano varieties and positively curved Kähler-Einstein metrics. The first result is a decomposition theorem for $\mathbb{Q}$-Fano varieties admitting a singular Kähler-Einstein metric. It states that up to a finite quasi-étale cover, any such variety is a product of varieties whose tangent sheaf is stable with respect to the anticanonical polarization. Along the way, we show the polystability of the so-called canonical extension of the tangent sheaf for such varieties, which is an important ingredient in the numerical characterization of Kähler-Einstein $\mathbb{Q}$-Fano varieties uniformized by the projective space. The second and last result of the section states that on a Fano manifold endowed with a smooth anticanonical divisor, the Kähler-Einstein metric with positive curvature and cone angle along the divisor collapses to an interval when the cone angle goes to zero, while the complete Ricci-flat Tian-Yau metric on the complement of the divisor occurs as a bubble.

The final part offers some open problems to explore in the future, ranging from decomposition and uniformization theorems in the negative curvature case to degeneration of conic Kähler-Einstein metrics with positive curvature.

PART I. NEGATIVE CURVATURE

1. Introduction

If X is a compact Kähler manifold such that K_X is ample, a celebrated theorem due independently to Aubin and Yau [**Aub78, Yau78**] asserts that there exists a unique Kähler-Einstein metric ω; i.e.

$$\operatorname{Ric}\omega = -\omega.$$

As an immediate application of this result and the Chern number computations of Chen-Ogiue [**CO75**], one obtains the so-called Miyaoka-Yau inequality as well as a uniformization result

Theorem 1.1. — *Let X be a compact Kähler manifold of dimension n such that K_X is ample. Then*

$$(2(n+1)c_2(X) - nc_1(X)^2) \cdot c_1(K_X)^{n-2} \geqslant 0, \tag{1.1}$$

and equality holds if and only if X is a ball quotient; i.e. $X = \mathbb{B}^n/\Gamma$ where $\Gamma \subset \operatorname{Aut}(\mathbb{B}^n)$ is a discrete subgroup acting freely.

The inequality is referred to as Miyaoka-Yau's inequality since in his celebrated result on the so-called generic semi-positivity of cotangent sheaves, Miyaoka proved that for any non-uniruled projective manifold, the inequality

$$\big(3c_2(X) - c_1^2(X)\big) \cdot H^{n-2} \geqslant 0, \tag{1.2}$$

holds, for any ample divisors $H \subset X$, cf. [**Miy87**]. When $\dim X = 2$ and $H = K_X$, this is nothing but the former inequality. In higher dimension, this inequality is both weaker when $H = K_X$ and stronger since it can apply to any minimal manifold for instance.

Another immediate application of Aubin-Yau result relies on the fact that if (X, ω) is Kähler-Einstein, then the hermitian metric h_ω on T_X induced by ω is actually Hermite-Einstein; i.e.

$$\Theta(T_X, h_\omega) \wedge \omega^{n-1} = \lambda \mathrm{Id}_{T_X} \otimes \omega^n$$

where $\lambda \in \mathbb{R}$ is a cohomological constant depending only on $c_1(X)$ and $[\omega] \in H^2(X, \mathbb{R})$. Then, the "easy" direction of the Kobayashi-Hitchin correspondence, usually attributed to Kobayashi and Lübke, implies the following:

Theorem 1.2. — *Let X be a compact Kähler manifold of dimension n such that K_X is ample and let ω be the Kähler-Einstein metric of X. Then, T_X is polystable with respect to K_X. More precisely, one can decompose*

$$T_X = \bigoplus_{i \in I} E_i$$

as a direct summand of K_X-stable subbundles with slope $-\frac{1}{n}(K_X^n)$ and which are parallel and mutually orthogonal with respect to h_ω.

In a different direction, Schumacher studied in [**Sch12**] smooth projective families $f : X \to Y$ such that K_X is relatively ample, or equivalently such that the fibers X_y have positive canonical bundle. If n is the relative dimension of f, the Kähler-Einstein metric ω_y constructed by Aubin and Yau yields a smooth hermitian metric h on $K_{X/Y}$ which is given by the expression ω_y^n on X_y via the identification $K_{X/Y}|_{X_y} \simeq K_{X_y}$. In particular, the curvature $\omega := i\Theta(K_{X/Y}, h)$ of h is a smooth, closed $(1,1)$-form such that $\omega|_{X_y} = \omega_y$ for every $y \in Y$ and it is natural to call it the relative Kähler-Einstein metric of the family. In that context, Schumacher proves the following

Theorem 1.3 **([Sch12]).** — *Let $f : X \to Y$ be a family of canonically polarized manifolds and let ω be the relative Kähler-Einstein metric. Then ω is semipositive and ω is positive along each fiber X_y such that the Kodaira-Spencer map $\kappa(y) : T_{Y,y} \to H^1(X_y, T_{X_y})$ is injective.*

One of the main relatively recent advances in the minimal model program [**BCHM10**] is the so-called finite generation of the canonical ring. That is, the canonical ring $R(X) = \oplus_{m=0}^{+\infty} H^0(X, mK_X)$ attached to a projective manifold is finitely generated. One of the most striking consequence is that any manifold X of general type (i.e. such that K_X is big) admits a birational model

$$X \dashrightarrow X_{\text{can}}$$

such that X_{can} has canonical singularities and $K_{X_{\text{can}}}$ is ample. It is unique (it is necessarily isomorphic to $\text{Proj}(R(X))$) and called the canonical model of X.

In this context, Aubin-Yau theorem has been generalized to singular varieties with canonical singularities (or even klt singularities) by Eyssidieux-Guedj-Zeriahi [**EGZ09**] (see also [**BEGZ10**] for the extension of their result to big cohomology classes):

Theorem 1.4 **([EGZ09]).** — *Let X be a projective manifold with klt singularities and ample canonical bundle. There exists a unique closed, positive current with local potentials $\omega \in c_1(K_X)$ such that*

(i) *ω is a genuine Kähler metric on X_{reg} and it satisfies $\text{Ric}\,\omega = -\omega$ on that locus.*
(ii) *$\int_{X_{\text{reg}}} \omega^n = (K_X^n)$.*

Note that the second condition (once the first one is satisfied) can be shown to be equivalent to the local potentials of ω being bounded.

2. Polystability of the tangent sheaf

In [**Eno88**], Enoki shows that the tangent sheaf of a projective variety X with canonical singularities and such that K_X is nef and big is semistable with respect to K_X. He used an approximation process for the singular Kähler-Einstein metric of X (which was not known to exist yet!) and relied on Yau's C^2-estimate to deal with Monge-Ampère equation with degenerate right-hand side.

Thanks to more robust pluripotential techniques and the existence of a singular Kähler-Einstein metric, we can actually fully generalize to the singular setting. More precisely, we proved

Theorem 2.1 (**[Gue16]**). — *Let X be a projective variety with klt singularities and ample canonical bundle, and let ω be the singular Kähler-Einstein metric. Then, the tangent sheaf T_X is polystable with respect to K_X. More precisely, one can decompose*

$$T_X = \bigoplus_{i \in I} E_i$$

as a direct summand of K_X-stable subsheaves with slope $-\frac{1}{n}(K_X^n)$ and which are parallel and mutually orthogonal with respect to h_ω on X_{reg}.

In the result above, we have denoted by h_ω the Hermitian metric induced on $T_{X_{\mathrm{reg}}}$. The result is actually proved to hold more generally for log canonical pairs (X, D) with D reduced, relying on the existence of a singular Kähler-Einstein metric obtained in [**BG14**] in this generalized context. Applications to stable varieties are also provided in [**Gue16**].

The proof of Theorem 2.1 above follows the same strategy as Enoki's proof, but in this more general context it uses the full force of Kołodziej's celebrated L^∞-estimate [**Koł98**]. Surprizingly, nothing more than the boundedness of the potentials are required in order to derive the semistability statement. In order to prove the polystability (and even just state the parallelism of each factor), we however need the existence and generic smoothness of singular Kähler-Einstein metrics, i.e. Theorem 1.4.

Relying on the above result, a theory of Higgs sheaves for klt varieties and a generalized Simpson's correspondence on such spaces, Greb-Kekebus-Peternell-Taji were able to generalize Theorem to the singular setting

Theorem 2.2 (**[GKPT19, GKPT20]**). — *Let X be a projective variety with klt singularities of dimension n such that K_X is ample. Then*

$$(2(n+1)\widehat{c}_2(X) - n\widehat{c}_1^2(X)) \cdot c_1(K_X)^{n-2} \geqslant 0,$$

and equality holds if and only if X is singular a ball quotient; i.e. $X = \mathbb{B}^n/\Gamma$ where $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ is a discrete subgroup acting freely in codimension one.

Let us comment briefly on the Chern classes $\widehat{c}_i$ in the statement above. On a singular projective variety, it is not straightforward to make sense of the Chern classes $c_i(X) = c_i(T_X)$ since T_X needs not be locally free anymore. Actually a conjecture of Zariski and Lipman asserts that T_X is locally free if and only if X is regular; the conjecture is known for a few classes of singularities including log terminal singularities. Several generalizations of Chern classes exist (e.g. due to Schwartz-MacPherson) but they do not coincide in general.

Mumford [**Mum83**] observed that given a coherent sheaf $\mathcal{E}$ on a projective variety X admitting a open subset X° with only finite quotient singularities, it is possible to define a cycle $\widehat{c}_i(\mathcal{E}) \in A^i(X) \otimes \mathbb{Q}$ whenever $i \leqslant \mathrm{codim}(X \setminus X^\circ) - 1$. Moreover, if X itself has only finite quotient singularities, then these Chern classes correspond to the orbifold Chern classes (after identifying the relevant homology and cohomology spaces), and if $S = D_1 \cap \cdots \cap D_{n-2}$ is a general complete intersection surface where $D_i \in |H|$ are ample, then S has quotient singularities and $\widehat{c}_2(\mathcal{E}) \cdot H^{n-2} = \widehat{c}_2(\mathcal{E}|_S)$ coincides with the usual second orbifold Chern number on the orbifold surface S.

Since varieties with klt singularities have at most finite quotients singularities in codimension two, it is then possible to define $\widehat{c}_2(X)$ and $\widehat{c}_1^2(X)$ which should be thought of as multilinear forms on $H^2(X,\mathbb{R})$, or as number once paired with H^{n-2} for some Cartier divisor H.

The derivation of Miyaoka-Yau's inequality from Theorem 2.1 is essentially a consequence of Simpson's approach to the MY inequality. More precisely, Simpson observed that the MY discriminant $\Delta_{\mathrm{MY}}(X) = 2(n+1)c_2(X) - nc_1(X)^2$ corresponds to the Bogomolov-Gieseker discriminant $\Delta_{\mathrm{BG}}(E) = 2rc_2(E) - (r-1)c_1(E)^2$ for a rank r vector bundle E applied to $E = T_X \oplus \mathcal{O}_X$. Moreover, if T_X is semistable with respect to K_X, then there exists a simple, explicit Higgs field $\theta : E \to E \otimes \Omega_X^1$ such that the Higgs bundle (E,θ) is K_X-stable. The Higgs version of Bogomolov-Gieseker inequality for stable bundles implies $\Delta_{\mathrm{BG}}(E) \cdot K_X^{n-2} \geqslant 0$; hence the same holds for $\Delta_{\mathrm{MY}}(X) \cdot K_X^{n-2}$.

In the singular case, this line of argument carries through essentially unchanged and allows one to obtain Miyaoka-Yau's inequality from the (poly)stability of T_X.

3. Miyaoka-Yau inequality for minimal models

Generalizations of Miyaoka-Yau inequalities have attracted a lot of attention over the last thirty years, with major contributions due to Tsuji, R. Kobayashi, Tian-Yau, Simpson, Megyesi, Y. Zhang, Song-Wang, Greb-Kebekus-Peternell-Taji (cf paragraph above) to cite only a few. We refer to the introduction of [**GT22**] for a more detailed account of this rich history.

In a joint work with Behrouz Taji, we have obtained a Miyaoka-Yau type inequality valid in a singular setting for minimal varieties X; that is when K_X is nef. One should think of that inequality as bridging the gap between the two inequalities of Miyaoka (1.2) and Yau (1.1), when the polarization is chosen to be the canonical one.

Theorem 3.1 **([GT22]).** — *Let X be a normal, projective variety of dimension n with klt singularities such that K_X is nef. Let ν denote the* numerical Kodaira dimension *of K_X. Then, for any ample divisor H in X, the inequality*

$$\big(2(n+1)\cdot \widehat{c}_2(X) - n\cdot \widehat{c}_1^{\,2}(X)\big)\cdot (K_X)^i \cdot H^j \geqslant 0. \tag{3.1}$$

holds, where $i = \min(\nu, n-2)$ and $j = n-i-2$.

We recall that the Chern classes above have to be understood in the sense of Mumford, and that the numerical Kodaira dimension $\nu(B)$ of a nef $\mathbb{Q}$-divisor B is defined by $\nu(B) := \max\{m \in \mathbb{N} \mid c_1(B)^m \neq 0\}$. In [**GT22**], we prove a more general inequality holding for dlt pairs with standard coefficients. In particular, it encompasses log smooth, log canonical pairs (X,D) with $K_X + D$ nef.

The proof of Theorem 3.1 follows a very simple strategy, but which unfortunately turns out to be quite technical. The idea is to first prove a Miyaoka-Yau type inequality for pairs (X,D) such that $K_X + D$ is ample, and then apply it to $D = \varepsilon H$ for some ample divisor H and $\varepsilon \to 0$ a small rational number.

The strategy itself therefore relies on being able to define classes like $\widehat{c}_2(X,D)$ for certain klt (or even dlt) pairs. We did it by generalizing Mumford's construction; a lot of technical difficulties appear because of the presence of a boundary divisor. Indeed, the local covers needed to work with necessarily have ramification in codimension one (along the support of the boundary divisor). As for global covers needed to derive geometric information, these will also ramify in codimension one along another divisor. This is the source of many complications, and we refer to [**GT22**] for more details about these difficulties and their treatment.

Once the relevant orbifold objects are defined and showed to satisfy relevant compatibility conditions, the next step is to to work with orbifold tangent bundles associated to an arbitrary dlt pair (X, D). The Kähler-Einstein techniques introduced in [**Gue16**] are robust enough to accomodate boundaries (this amounts to introducing cone-like of Poincaré-like singularities) and thus prove that the orbifold tangent sheaf of a log canonical pair (X, D) such that $K_X + D$ is ample (or nef and big) is semistable with respect to K_X.

Then, one needs to adapt Simpson's argument to derive a MY-type inequality for such pairs (X, D). This turns out to be quite tricky because of the presence of singularities and the quite complex definition of Chern classes like $\widehat{c}_2(X, D)$. One significant difference compared to e.g. [**GKPT19**] is that the local uniformizing charts are not quasi-étale anymore, but they ramify along the boundary divisor D. In particular, we cannot allow arbitrary coefficients on the boundary divisor as we might otherwise leave the klt (even lc) world as soon as one performs the necessary task of taking Kawamata cyclic covers in order to clear the denominators in D.

Finally, with the MY inequality in hand for the pair $(X, \varepsilon H)$, one needs to pass to the limit when $\varepsilon \to 0$. This harmless process in the smooth (or log smooth) context becomes delicate in the presence of singularities, again. But it can be achieved!

4. Kähler manifolds with negative holomorphic sectional curvature

4.1. The compact case. — Let M be a compact Kähler manifold of dimension n and let ω be a Kähler metric on M such that its holomorphic sectional curvature is negative; that is, for every $x \in M$ and any $[v] \in \mathbb{P}(T_{M,x})$, one has $\mathrm{HSC}_\omega(x, [v]) < 0$.

Let us briefly recall how that last quantity is defined. Start by picking a point $x \in X$ and a system of holomorphic coordinates (z_i) near x, assumed to be orthonormal at x. If $(R_{i\bar{j}k\bar{\ell}})$ is the curvature tensor of ω in these coordinates and if $v = \sum v_i \frac{\partial}{\partial z_i}$ is a non-zero tangent vector at x, then the holomorphic sectional curvature of (M, ω) at $(x, [v])$ is defined by

$$\mathrm{HSC}_\omega(x, [v]) := \frac{1}{|v|_\omega^4} \cdot \sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}} v_i \bar{v}_j v_k \bar{v}_\ell.$$

Alternatively, one can define $\mathrm{HSC}_\omega(x, [v])$ as the supremum of the Gaussian curvature of $(\Delta, f^*\omega)$ at the origin among all immersed holomorphic disks $f : \Delta \to X$ such that $f(0) = x$ and $df(\frac{\partial}{\partial t}) = v$.

Under the assumptions on (M, ω) above, the Ahlfors-Schwarz lemma shows that M is Brody hyperbolic; that is, every holomorphic map $f : \mathbb{C} \to M$ is constant. Hyperbolicity for projective (or merely compact Kähler) manifolds is conjectured to be related to algebraic properties. More precisely, S. Lang formulated the following

Conjecture 4.1. — *A projective manifold X is hyperbolic if and only if each of its subvarieties (including X itself) is of general type.*

Recall that an irreducible projective variety Y is said to be of general type if the canonical bundle $K_{\widetilde{Y}}$ of any smooth birational model $\widetilde{Y}$ of Y is big; that is, $\widetilde{Y}$ has maximal Kodaira dimension. More than thirty years after its formulation, Lang's conjecture remains mostly open. Besides the trivial case of curves, the known cases of the conjecture are:

- Surfaces with some specific geometry [**Des79, GG80, MM83, McQ98**].
- Generic hypersurfaces of high degree in $\mathbb{P}^n$.
  By the work of Clemens [**Cle86**], Ein [**Ein88, Ein91**] and Voisin [**Voi96**] later improved by Pacienza [**Pac04**], their subvarieties are of general type. Moreover, they are hyperbolic thanks to the recent breakthroughs by Siu [**Siu15**] and Brotbek [**Bro17**] independently; cf also Demailly [**Dem18**].
- Quotients of bounded domains (Boucksom and Diverio [**BD21**]).

Let us go back to the case of a compact Kähler manifold (M, ω) with negative holomorphic sectional curvature. It was proved by Wu and Yau [**WY16**] that K_M is ample provided that M is a projective manifold. Shortly after, Tosatti and Yang [**TY17**] extended the result to the general Kähler case. In particular, under those general assumptions, M is automatically projective. Now, if $Y \subset M$ is a *smooth* subvariety of M, then the decreasing property of the holomorphic (bi)sectional curvature shows that K_Y is ample again. However, in view of Lang's conjecture, it is crucial to control the geometry of *singular* subvarieties of M as well. In [**Gue22**], we proved the following

Theorem 4.2 **([Gue22]).** — *Let (M, ω) be a compact Kähler manifold with negative holomorphic sectional curvature and let $Y \subseteq M$ be a possibly singular, irreducible subvariety of M. Then, Y is of general type.*

An an immediate application, we get

Corollary 4.3. — *Lang's conjecture holds for compact manifolds M admitting a Kähler metric with negative holomorphic sectional curvature.*

The main original idea is to construct on a desingularization $\widetilde{Y}$ of Y a family of singular Kähler-Einstein metrics $(\omega_b)_{b>0}$ having generically cone singularities along a given ample divisor B and whose cone angle $2\pi(1-b)$ is meant to tend to 2π. These metrics are relatively well-understood only on the log canonical model of $(\widetilde{Y}, bB)$ and the heart of the proof consists in working on these varying birational models and to show that the volume of ω_b does not go to 0 when b approaches 0. The general idea of using a continuity method and Royden's Laplacian estimate originates from [**WY16**], but the degree of technicality in the singular setting is significantly higher. For instance, the Ricci curvature blows down to $-\infty$, thus prohibiting the use of a maximum principle. Also, as the computations are performed on spaces which depend on the parameter b, establishing the volume estimate requires a delicate analysis.

We should mention that Theorem 4.2 generalizes to the case of quasi-negative holomorphic sectional curvature, where one needs to use as an important first step a result of Diverio-Trapani [**DT19**].

4.2. The quasi-projective case. — Another way to think of the situation of Theorem 4.2 is to view Y_{reg} as a quasi-projective manifold endowed with a Kähler metric ω such that

- ω has negative holomorphic sectional curvature;
- ω extends smoothly to a (singular) compactification.

Given this point of view, it is natural to ask to which extent Theorem 4.2 generalizes to arbitrary quasi-projective manifolds. More precisely, given a projective manifold X, a reduced divisor D with simple normal crossings and a Kähler metric ω on $X^\circ := X \backslash D$ with negative holomorphic curvature, is it true that (X, D) is of log general type; that is, $K_X + D$ is big?

This question is in part motivated by results of Cadorel [**Cad21**] who proved that given a projective log smooth pair (X, D) such that X° admits a Kähler metric ω with negative holomorphic sectional curvature and non-positive holomorphic *bisectional* curvature, then $\Omega_X(\log D)$ is big, and, moreover, Ω_X is big provided that ω is bounded near D.

His proof involves working on $\mathbb{P}(\Omega_X(\log D))$ and considering the tautological line bundle $\mathcal{O}(1)$ on it. By the assumption on the *bisectional* curvature, ω induces a smooth, non-negatively curved hermitian metric h on $\mathcal{O}(1)$ away from (the inverse image of) D. Moreover, the Alhfors-Schwarz lemma guarantees that h extends across D as a singular metric with non-negative curvature. Using a result of Boucksom [**Bou02**] on a metric characterization of bigness then completes the proof.

One cannot expect such a strong result on the logarithmic cotangent bundle if one drops the assumption on the bisectional curvature. However, it seems reasonable to expect it for the logarithmic canonical bundle. The main difficulty is that one does not get from ω a positively curved metric on $K_X + D$ even on a Zariski open set. So one has to produce such a metric out of other methods, like the continuity method, cf [**WY16**]. However, one faces several new difficulties compared to the setting of Theorem 4.2:

$\rightarrow$ To start the continuity method, one needs $K_X + D$ to be pseudo-effective. In the case $D = 0$, this is a consequence of the absence of rational curves (Ahlfors-Schwarz lemma) combined with Mori's bend and break and [**BDPP13**]. If D is not empty then one only knows that X° has no entire curves hence X has no rational curve meeting D at at most two points. To conclude, one would then need to have a logarithmic version of Mori's bend and break, but unfortunately it is not known as of now. To circumvent the difficulty and inspired by the proof of [**CP15**, Thm. 4.1], we modify the boundary D into $D + sB$ for some ample B and some $s > 0$ to make $K_X + D + sB$ psef. Only at the very end of the argument, one will see that $K_X + D$ is pseudoeffective.

$\rightarrow$ The finiteness of the log canonical ring, known for klt pairs and crucial to understanding the deforming Kähler-Einstein metrics, is not known for lc pairs like (X, D). The idea is then to deform (X, D) into a klt pair $(X, \Delta_{b,s} := (1-b)D + (b+s)B)$ that makes it klt and of log general type.

Give or take these adjustements, one can still run the strategy of Theorem 4.2 *mutatis mutandis*. A very important point is that the behavior of ω near D is not arbitrary, as ω must be dominated by a metric with Poincaré singularities along D thanks to Ahlfors-Schwarz lemma. In the end, the result is the following

Theorem 4.4 **([Gue22]).** — *Let (X, D) be a pair consisting of a projective manifold X and a reduced divisor $D = \sum_{i\in I} D_i$ with simple normal crossings. Let ω be a Kähler metric on $X^\circ := X\backslash D$ such that there exists $\kappa_0 > 0$ satisfying*

$$\forall(x, v) \in X^\circ \times T_{X,x}\backslash\{0\}, \quad \mathrm{HSC}_\omega(x, [v]) < -\kappa_0.$$

Then, the pair (X, D) is of log general type; that is, $K_X + D$ is big. If additionnally ω is assumed to be bounded near D, then K_X is big.

5. Families of manifolds of general type

If $f : X \to Y$ is a projective family of canonically polarized manifolds (say over a smooth base), then we have seen in Theorem 1.3 that the relative Kähler-Einstein metric $\omega \in c_1(K_{X/Y})$ is a semipositive form. Schumacher's proof consists in applying the minimum principle to the following equation

$$\Delta_{\omega_y} c(\omega) = -c(\omega) + |\bar\partial v|^2_\omega$$

holding on any fixed given fiber X_y, and where

- a one-dimensional disk $\mathbb{D} \subset Y$ through y is given, and f is co-restricted to $\mathbb{D}$.
- $v \in \mathcal{C}^\infty(X, T_X^{1,0})$ is the horizontal lift of $\frac{\partial}{\partial t}$ with respect to ω, as introduced by Siu. In particular, $df(v) = \frac{\partial}{\partial t}$.
- $c(\omega)$ is the geodesic curvature of ω, i.e. $c(\omega) = \frac{\omega^{n+1}}{\omega^n\wedge f^*(idt\wedge d\bar t)}$.

A couple of years later, Tsuji gave a completely different proof of the semipositivity of the relative Kähler-Einstein metric by relying on the following foundational result of Berndtsson and Păun:

Theorem 5.1 **([BP08]).** — *Let $f : X \to Y$ be a surjective projective map between smooth manifolds, and let (L, h) be a holomorphic line bundle endowed with a metric such that*

(1) The curvature current of (L, h) is semipositive on X; i.e. $\Theta(L, h) \geqslant 0$.
(2) $H^0(X_y, (K_{X_y} + L) \otimes \mathscr{I}(h)) \neq 0$ for some $y \in Y^\circ$.

Then the relative Bergman kernel metric of the bundle $K_{X/Y} + L|_{X^\circ}$ is not identically $-\infty$. It has semipositive curvature current and extends across $X \setminus X^\circ$ to a metric with positive curvature on X.

Here, Y° is the complement of the locus of critical values of f, and $X^\circ = f^{-1}(Y^\circ)$.

Coming back to families of canonically polarized manifolds $f : X \to Y$, Tsuji [**Tsu10**] defined a sequence of metrics h_m on $mK_{X/Y} + A = K_{X/Y} + \underbrace{(m-1)K_{X/Y} + A}_{L}$ where A is some fixed very ample line bundles as the Bergman kernel metric associated to $(L, h_{m-1} \otimes h_A)$. He shows that there exists an a priori singular hermitian metric h_∞ on $K_{X/Y}$ which is the weak limit

$$m!^{\frac{n}{m}} h_m^{\frac{1}{m}} \longrightarrow h_\infty$$

and moreover $\Theta(K_{X/Y}, h_\infty)|_{X^\circ}$ coincides with the relative Kähler-Einstein metric ω of f. In particular $\omega \geqslant 0$ by [**BP08**] and it extends to a positive current in $c_1(K_{X/Y})$ across $X \setminus X^\circ$.

In view of [**BP08**] and Theorem 1.3, it is natural to expect that relative *singular* Kähler-Einstein metrics with negative curvature should also vary in a plurisubharmonic way. More precisely, if $f : X \to Y$ is a projective family such that K_{X_y} is big for $y \in Y^\circ$, there exists a unique singular Kähler-Einstein metric $\omega_y \in c_1(K_{X_y})$ thanks to [**BCHM10**] and Theorem 1.4 and they glue to a canonical current ω on X° in the same way as in the canonically polarized case. Base points and poor regularity of ω_y near $\mathbb{B}_+(K_{X_y})$ (and all the more for ω) make Schumacher's approach essentially hopeless in this singular situation. At first sight, the method of Tsuji looks very general. However, it uses in an essential manner the asymptotic expansion of Bergman kernels, which depends on at least two derivatives of the metric. Using a quite intricate regularization process, we managed to show that Tsuji's method can still be carried through for families of manifolds of general type:

Theorem 5.2 **([CGP21]).** — *Let $f : X \to Y$ be a projective family of manifolds of general type, and let $\omega \in c_1(K_{X^\circ/Y^\circ})$ be the relative Kähler-Einstein metric. Then, ω is a weak limit of relative Bergman kernels, and as such, it is a positive current and it extends to a positive current in $c_1(K_{X/Y})$.*

The result is actually proved in a slightly larger degree of generality for Kähler-Einstein metrics associated to pairs $(X, (L, h))$ such that $K_X + L$ is big and where (L, h) is an hermitian line bundle such that $\Theta(L, h) \geqslant 0$ and h has zero Lelong numbers everywhere on X. Unfortunately, our methods come short of reaching the general case of klt pairs (X, B) with $K_X + B$ big. As another consequence of the theorem above, we can derive a precise estimate at the level of potentials for the singular relative Kähler-Einstein metric, cf [**DGG20**, Remark 5.6]

6. From cones to cusps

Let X be a complex projective manifold and let $D \subset X$ be a smooth divisor, and set $X^\circ := X \setminus D$. Recall that if $\beta \in (0, 1)$, a Kähler metric ω on X° is said to have cone singularities along D with cone angle $2\pi\beta$ if it is locally quasi-isometric to the model cone metric

$$\omega_{\beta,\mathrm{mod}} := \frac{idz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{j \geqslant 2} idz_j \wedge d\bar{z}_j$$

on each coordinate chart $(U, (z_i))$ where $U \cap D = (z_1 = 0)$. Such a metric is incomplete, has finite volume and automatically extends to a closed, positive $(1, 1)$-current on X. There is an analogue of the Aubin-Yau theorem guaranteeing the existence and uniqueness of a negatively curved Kähler-Einstein metric ω_β with cone angle $2\pi\beta$ along D under the condition that the adjoint $\mathbb{R}$-line bundle $K_X + (1-\beta)D$ is ample, cf e.g. [**Bre13, CGP13, GP16, JMR16**]. Such a metric will then be solving the equation

$$\mathrm{Ric}\,\omega_\beta = -\omega_\beta + (1-\beta)[D] \tag{6.1}$$

understood in the sense of currents.

6.1. The case K_X+D ample. — If one assumes that K_X+D is ample, then the same will hold true for $K_X+(1-\beta)D$ as long as β is small enough. In that situation, we have at hand several Kähler-Einstein metrics on X°. First, the (incomplete) KE cone metrics ω_β, and then the unique complete KE metric ω constructed by R. Kobayashi [**Kob84**] and Tian-Yau [**TY87**]. Recall that ω has Poincaré growth near D, meaning that it is locally quasi-isometric to

$$\frac{idz_1\wedge d\bar z_1}{|z_1|^2\log^2|z_1|^2}+\sum_{j\geqslant 2} idz_j\wedge d\bar z_j$$

whenever D is given by $(z_1=0)$ in some coordinate chart. The connection between these metrics was given by

Theorem 6.1 **([Gue20b]).** — *Let X be a projective manifold and let D be a smooth divisor such that K_X+D is ample. Then the Kähler-Einstein cone metrics ω_β converge to the complete Kähler-Einstein metric ω on $X\setminus D$ when $\beta\to 0$. The convergence happens both weakly as currents on X and locally smoothly on $X\setminus D$.*

Actually, we gave an explicit model metric

$$\widehat{\omega}_\beta := \frac{\beta^2 idz_1\wedge d\bar z_1}{|z_1|^{2(1-\beta)}(1-|z_1|^{2\beta})^2}+\sum_{k=2}^n idz_k\wedge d\bar z_k$$

such that there is a constant $C>0$ independent of β satisfying

$$C^{-1}\widehat{\omega}_\beta\leqslant \omega_\beta\leqslant C\widehat{\omega}_\beta.$$

The metric $\frac{\beta^2 idz_1\wedge d\bar z_1}{|z_1|^{2(1-\beta)}(1-|z_1|^{2\beta})^2}$ is actually the natural Kähler-Einstein cone metric on $\mathbb{D}^*$ with cone singularity at the origin, as one can convince oneself by choosing $\beta=\frac{1}{m}$ and pushing down the Poincaré metric on $\mathbb{D}$ via the map $z\mapsto z^m$. On that model, it is clear to see how the cone singularity degenerates to the cusp when the angle goes to zero.

Interestingly enough, this quasi-isometric control of ω_β also allowed us to show that if $p\in D$, the rescaled metrics $(X,\beta^{-2}\omega_\beta,p)$ converge in pointed Gromov-Hausdorff sense to a cylindrical metric on $\mathbb{C}^*\times\mathbb{C}^{n-1}$, up to extracting a sequence (β_k), cf [**Gue20b**, Theorem 6.2].

6.2. Closing the cusp. — Another interesting example is provided by toroidal compactifications of ball quotients $X^\circ=\Gamma\backslash\mathbb{B}^n$, where $\Gamma\subset \mathrm{Aut}(\mathbb{B}^n)$ is a torsion-free, discrete subgroup. It is well-known that one can embed $X^\circ\hookrightarrow X$ as a Zariski-open subset of a projective orbifold X such that $D:=X\setminus X^\circ$ is a disjoint union of abelian varieties. Note that the Bergman metric on $\mathbb{B}^n$ descends to the complex (complete) hyperbolic metric $\omega_{\rm hyp}$ on X°, which we normalize to have $\mathrm{Ric}\,\omega_{\rm hyp}=-\omega_{\rm hyp}$. Capitalizing on the fact that D admits an euclidean neighborhood $U\subset X$ which is isomorphic to a neighborhood of the zero section in the normal bundle $N_{D|X}$ of D in X, one can get an exact and very explicit description of $\omega_{\rm hyp}$ on U, cf. e.g. [**Mok12**, Eq. (8)]. As a byproduct, one sees that if $(z_1,\dots,z_n)$ is a system of holomorphic coordinates on some open set $U\subset X$ such that $D \cap U = (z_1 = 0)$, then $\omega_{\rm hyp}|_U$ is quasi-isometric to

$$\frac{idz_1 \wedge d\bar z_1}{|z_1|^2(-\log|z_1|)^2} + \frac{1}{(-\log|z_1|)}\sum_{k=2}^n idz_k \wedge d\bar z_k.$$

As opposed to a Poincaré type metric, one notices the degenerating factor $\frac{1}{(-\log|z_1|)}$ in front of the tangential directions of D which explain that $\omega_{\rm hyp}$ contracts D.

This can be interpreted as well via the map to the singular, minimal compactification of Sataka-Baily-Borel $X_{\rm min}$ of X°. Indeed, given the universal property it satisfies (any normal compactification $X^\circ \hookrightarrow \overline{X}$ yields a unique morphism $\overline{X} \to X_{\rm min}$), we get a birational morphism $\pi : X \to X_{\rm min}$ and one can check that $K_X + D = \pi^* K_{X_{\rm min}}$. From this, it is easy to see that $K_X + (1-\beta)D$ is ample for $0 < \beta \ll 1$ (but certainly not for $\beta = 0$ unless $D = \emptyset$) and therefore X° also comes equipped with KE metrics ω_β with cone angle $2\pi\beta$ along D whenever $\beta > 0$ is small enough. Again, we can find an explicit relationship between these metrics thanks to the following

Theorem 6.2 **([BG22]).** — *Let (X, D) be a toroidal compactification of a ball quotient $X^\circ = \Gamma \backslash \mathbb{B}^n$, and let ω_β be the KE metric solving* (6.1) *for small β. Then, we have convergence*

$$\omega_\beta \xrightarrow[\beta \to 0]{} \omega_{\rm hyp}$$

both in $C^\infty_{\rm loc}(X^\circ)$ and weakly as currents on X. Moreover, we have precise asymptotics of ω_β near D when $\beta \to 0$.

On the asymptotics.

The asymptotics are obtained by constructing a model metric on the normal bundle of D using the Calabi Ansatz, and showing that its curvature is bounded uniformly in the cone angle. In order to give those, we need to introduce some notation. As explained above, one can work on a euclidean neighborhood of $D \subset X$ isomorphic to an open neighborhood U of the zero section in the total space of $L := N_{D|X}$. Since D is a torus, we can pick a flat metric $\theta_D \in c_1(L^*)$ on D and choose an hermitian metric h on L such that $i\Theta(L, h) = -\theta_D$. This yields a function $t := \log|v|_h^2$ on $L \setminus D$. We also have on $L \setminus D$ a connection 1-form η which coincides on each fibre of L with the angular form $d\theta$, and satisfies

$$d\eta = -ip^*\Theta(L, h) = p^*\theta_D,$$

where p is the projection $p : L \to D$. Then $\xi = \frac{1}{2}dt + i\eta$ is a (1,0)-form on $L \setminus D$, coinciding with $\frac{dz}{z}$ in each fibre. In particular, $dt \wedge \eta = i\xi \wedge \bar\xi$ coincides with $\frac{idz \wedge d\bar z}{|z|^2}$ in each fiber. In the following, one will identify $p^*\theta_D$ with θ_D and view the latter as a $(1,1)$-form on the total space L.

We can now describe the behavior or the Kähler-Einstein cone metric ω_β on U as β approaches zero:

- On $\{\beta t \to 0\}$, it is quasi-isometric to

$$\omega_{\rm KE} = (n+1)\Big[\frac{i\xi \wedge \bar\xi}{(-t)^2} + \frac{\theta_D}{-t}\Big]$$

  with quasi-isometry constant converging to 1 as $\beta t \to 0$.

- On $\{\beta t \to -\infty\}$, it is quasi-isometric to
$$a_n \beta^2 \cdot e^{\beta t} i\xi \wedge \bar{\xi} + \beta\theta_D$$
with quasi-isometry constant converging to 1 as $\beta t \to -\infty$ and $\beta \to 0$ and where a_n is some constant that can be made explicit via some integrals.

- Elsewhere, i.e. on $\{-C \leqslant \beta t \leqslant C^{-1}\}$; it is quasi-isometric to
$$\beta^2 \cdot e^{\beta t} i\xi \wedge \bar{\xi} + \beta\theta_D$$
with quasi-isometry constant uniformly bounded as $\beta \to 0$.

A few comments on the above asymptotics. One recovers from the first item that on the compact sets of X°, corresponding to $(t \geqslant -C)$, ω_β is asymptotic to $\omega_{\rm hyp}$.

Setting $r = e^{\frac{\beta t}{2}}$, we have $\beta^2 \cdot e^{\beta t} i\xi \otimes \bar{\xi} \simeq dr^2 + \beta^2 r^2 \eta^2$. This implies that on $(\beta t \leqslant C^{-1}) = (r \leqslant e^{\frac{1}{2C}})$, the divisor is collapsed at speed $\sqrt{\beta}$ while the circle directions are collapsed at speed β. We will observe a similar phenomenon in a different geometric setting (positively curved KE cone metric associated to a smooth anticanonical divisor D in a Fano manifold).

Bounded symmetric domains.
The first half of the statement (i.e. the convergence part) remains true in the more general setting of quotients of bounded symmetric domains, cf [**BG22**, Theorem 1.1]. In that case, D needs not be smooth anymore but has simple normal crossings up to a finite cover.

Closing the cusp.
Assume that the lattice Γ is arithmetic. By choosing the angles carefully along each torus at the boundary, one can find a sequence ω_{β_m} of orbifold KE metrics that can be globally desingularized so that $(X^\circ, \omega_{\rm hyp})$ is the limit of *smooth, compact* KE spaces up to the action of a larger and larger group of isometries, cf [**BG22**, § 4.3]. In a nutshell, one can "close the complex hyperbolic cusp".

PART II. ZERO CURVATURE

7. Introduction

7.1. Compact Kähler manifold with zero first Chern class. — Let (X,ω_0) be a compact Kähler manifold of dimension n such that $c_1(X)=0\in H^2(X,\mathbb{R})$. One of the most celebrated theorems of Yau [**Yau78**] gives the existence of a unique Kähler Ricci-flat metric $\omega\in[\omega_0]$; i.e.

$$\mathrm{Ric}\,\omega = 0.$$

Classical Chern classes computations (cf e.g. Chen-Ogiue [**CO75**]) show that for a Ricci flat Kähler metric ω, one has

$$c_2(X,\omega)\wedge\omega^{n-2} = b_n\|\Theta(T_X,\omega)\|^2_\omega\,\omega^n$$

for some dimensional constant $b_n>0$. In particular, integrating the identity over X, one gets

$$c_2(X)\cdot[\omega_0]^{n-2}\geqslant 0,$$

and the vanishing of that quantity implies that ω is flat. In turn, this implies that the Riemannian universal cover of (X,ω) is $(\mathbb{C}^n,\omega_{\mathrm{eucl}})$, and then, Bieberbach's theorem ensures that $X=T/G$ for some complex torus T and some finite group $G\subset\mathrm{Aut}(T)$ acting freely. All in all, we get

Theorem 7.1. — *Let X be a compact Kähler manifold of dimension n. The following assertions are equivalent*

(i) There exists a complex torus T and some finite group $G\subset\mathrm{Aut}(T)$ acting freely such that

$$X\simeq T/G.$$

(ii) We have $c_1(X)=0\in H^2(X,\mathbb{R})$, and there exists a Kähler class $\alpha\in H^2(X,\mathbb{R})$ such that

$$c_2(X)\cdot\alpha^{n-2}=0.$$

In the same vein as in the negative curvature case, Yau's result implies the following

Theorem 7.2. — *Let (X,ω_0) be a compact Kähler manifold of dimension n such that $c_1(X)=0$ let $\omega\in[\omega_0]$ be the Kähler Ricci-flat metric. Then, T_X is polystable with respect to $[\omega_0]$. More precisely, one can decompose*

$$T_X=\bigoplus_{i\in I}E_i$$

as a direct summand of $[\omega_0]$-stable subbundles with $c_1(E_i)=0$ and which are parallel and mutually orthogonal with respect to ω.

The structure of compact Kähler manifolds with vanishing first Chern class is encapsulated in the following fundamental result, which relies in an essential way on Yau's solution of the Calabi conjecture.

Theorem 7.3 **(Beauville-Bogomolov decomposition Theorem, [Bea83])**
Let X be a compact Kähler manifold with $c_1(X) = 0 \in H^2(X, \mathbb{R})$. Then, there exists a finite étale cover $\widetilde{X} \to X$ such that $\widetilde{X}$ decomposes as a Kähler manifold as follows,

$$\widetilde{X} = T \times \prod_i Y_i \times \prod_j Z_j,$$

where T is a complex torus, and where the Y_i (resp. Z_j) are irreducible and simply connected Calabi-Yau manifolds (resp. holomorphic symplectic manifolds).

The strategy of the proof of Theorem 7.3 consists in first using Yau's theorem in order to equip X with a Ricci-flat Kähler metric, and then applying the deep theorems of de Rham and Cheeger-Gromoll to split a finite étale cover of X according to its holonomy decomposition. The identification of the factors then follows from the Berger-Simons classification of holonomy groups combined with the Bochner principle, which states that holomorphic tensors are parallel.

7.2. Singularities and uniformization results in the projective case. — Conjecturally, any compact Kähler manifold X with $\kappa(X) = 0$ should admit a bimeromorphic minimal model

$$X \dashrightarrow X_{\min}$$

such that $X_{\min}$ has terminal singularities and $c_1(X) = 0 \in H^2(X, \mathbb{R})$. From this perspective, it is natural to try to extend the Beauville-Bogomolov decomposition theorem to the singular setting.

The first step in that direction was achieved by Eyssidieux-Guedj-Zeriahi [**EGZ09**]. Their argument for the generic smoothness required a projectivity assumption which was later removed by Păun [**Pău08**]:

Theorem 7.4 **([EGZ09, Pău08]).** — *Let (X, ω_0) be a compact Kähler variety with klt singularities and $c_1(X) = 0$. There exists a unique closed, positive current with local potentials $\omega \in [\omega_0]$ such that*

(i) *ω is a genuine Kähler metric on X_{reg} and it satisfies* $\operatorname{Ric}\omega = 0$ *on that locus.*
(ii) $\int_{X_{\text{reg}}} \omega^n = \int_X \omega_0^n$.

Note that the second condition (once the first one is satisfied) can be shown to be equivalent to the local potentials of ω being bounded.

An important step in our topological understanding of klt singularities was achieved by [**Xu14**] who proved that the regional fundamental group of a klt singularity is finite. Building upon that result, Greb-Kebekus-Peternell used a clever induction argument to prove the following

Theorem 7.5 **([GKP16b]).** — *Let X be a quasi-projective variety with klt singularities. There exists a finite, quasi-étale cover $\widetilde{X} \to X$ such that the inclusion $\widetilde{X}_{\text{reg}} \to \widetilde{X}$ induced an isomorphism between the étale fundamental groups*

$$\hat{\pi}_1(\widetilde{X}_{\text{reg}}) \xrightarrow{\sim} \hat{\pi}_1(\widetilde{X}).$$

Equivalenty, any holomorphically flat vector bundle on X_{reg} extends to a flat vector bundle on $\widetilde{X}$.

Given the solution of Zariski-Lipman conjecture for klt singularities [**Dru14, GK14, GKKP11**], an easy consequence of the above theorem is the following

Corollary 7.6 **([GKP16a]).** — *Let X be a projective variety with klt singularities such that $T_{X_{\mathrm{reg}}}$ is flat. Then, there exists an abelian variety A and a finite, quasi-étale Galois cover $A \to X$. In other words, X is the quotient of an abelian variety by a finite group acting freely in codimension one.*

One would like to mention the recent striking theorem of Braun about the local fundamental group of a klt singularity. Recall that if (X, x) is a germ of a complex variety, the fundamental groups of U_{reg} when U run over a fundamental system of neighborhoods of x eventually stabilizes, and one set $\pi_1(X_{\mathrm{reg}}, x) = \pi_1(U_{\mathrm{reg}})$ for U small enough in that sense.

Theorem 7.7 **([Bra21]).** — *Let (X, x) be a germ of a quasi-projective klt singularity. Then, the local fundamental group $\pi_1(X_{\mathrm{reg}}, x)$ is finite.*

Using Theorem 7.7, the proof of Theorem 7.5 can be drastically simplified.

Remark 7.8. — The algebraicity assumption in Theorem 7.7 may sound odd as the result is fundamentally dealing with the euclidean topology. However, in the analytic setting such a finiteness result for the local fundamental group is still unknown. However, and as explained in [**CGGN22**, Remark 6.10] the only missing step is to be able to run a relative MMP for *projective* morphisms between analytic (Stein) spaces. This has been achieved very recently by Fujino [**Fuj22**].

Let us end this introductory paragraph by mentioning the following uniformization result generalizing Theorem 7.1 to the singular projective case. It is due to Greb-Kebekus-Peternell when X is smooth in codimension two, and to Lu-Taji in full generality.

Theorem 7.9 **([GKP16a, LT18]).** — *Let X be a projective variety of dimension n with klt singularities. The following assertions are equivalent*

(i) There exists an abelian variety T and some finite group $G \subset \mathrm{Aut}(T)$ acting freely in codimension one such that
$$X \simeq A/G.$$
(ii) We have $c_1(X) = 0 \in H^2(X, \mathbb{R})$, and there exists an ample Cartier divisor H such that
$$\hat{c}_2(X) \cdot H^{n-2} = 0.$$

As in Theorem 2.2 and the remark below it, the Chern class $\widehat{c}_2$ refers to Mumford's Chern class. If X is smooth in codimension two and $f : Y \to X$ is a log resolution, then one has $\hat{c}_2(X) \cdot H^{n-2} = c_2(Y) \cdot (f^*H)^{n-2}$.

Strategy of proof of Theorem 7.9. — Assume for simplicity that X is smooth in codimension two. Pick a general complete intersection $S = D_1 \cap \ldots \cap D_{n-2}$ with $D_i \in |mH|$ for m large. Then $\mathscr{E} := T_X|_S$ is a locally free sheaf on the smooth surface S with vanishing first and second Chern class. Moreover, $\mathscr{E}$ is H-polystable thanks to [**GKP16b**] and Mehta-Ramanathan theorem (more precisely, Flenner's version of that result in the singular setting). By the Kobayashi-Hitchin correspondence, $\mathscr{E}$ is holomorphically flat, given by a unitary representation $\pi_1(S) \to \mathrm{U}(n)$.

By the singular version of Lefschetz hyperplane theorem, the inclusion $S \subset X_{\mathrm{reg}}$ yields an isomorphism $\pi_1(S) \simeq \pi_1(X_{\mathrm{reg}})$. In particular $\mathscr{E}$ is given by the restriction to S of a flat bundle $\mathscr{F}$ on X_{reg}. It remains to see that if S is general, the isomorphism $\mathscr{F}|_S \simeq T_X|_S$ implies that $\mathscr{F}$ is isomorphic to T_X over X_{reg}, so that $T_{X_{\mathrm{reg}}}$ is flat. The theorem now follows from Corollary 7.6. □

In the following sections, we will explain how the singular Kähler Ricci flat metrics constructed by Eyssidieux-Guedj-Zeriahi and the results of Greb-Kebekus-Peternell recalled above led to establishing a full generalization of Theorem 7.3 and Theorem 7.1 in the klt Kähler setting.

8. The Bochner principle and the holonomy cover

If X is a smooth compact Kähler manifold and if ω is a Kähler Ricci-flat metric, then Bochner formula states that any holomorphic tensor $\sigma \in H^0(X, T_X^{\otimes p} \otimes \Omega_X^{\otimes q})$ satisfies

$$\Delta_\omega |\sigma|^2_\omega = |\nabla \sigma|^2_\omega$$

where ∇ is the Chern connection induced by ω on $T_X^{\otimes p} \otimes \Omega_X^{\otimes q}$. In particular, integrating the formula above yields $\nabla\sigma \equiv 0$; i.e. σ is parallel. Said otherwise, σ is invariant under parallel transport. In particular, fixing a point $x \in X$ and setting $V := T_{X,x}$, we get a 1:1 correspondence between holomorphic tensors $\sigma \in H^0(X, T_X^{\otimes p} \otimes \Omega_X^{\otimes q})$ and invariant vectors $v \in (V^{\otimes p} \otimes (V^*)^{\otimes q})^G$ where $G = \mathrm{Hol}(X, \omega)$ is the holonomy group of G, whose action on $V^{\otimes p} \otimes (V^*)^{\otimes q}$ is the natural one obtained by tensorization of the action of G on V.

With Daniel Greb and Stefan Kebekus, we generalized this correspondence in the projective klt case, and later with Benoît Claudon, Patrick Graf and Philipp Naumann, we proved the general klt Kähler case:

***Theorem 8.1* (Bochner principle, [GGK19, CGGN22])**
Let (X, ω_0) be a compact Kähler variety with klt singularities and $c_1(X) = 0$, and let $\omega \in [\omega_0]$ be the singular Ricci flat metric. Then, any holomorphic tensor $\sigma \in H^0(X_{\mathrm{reg}}, T_X^{\otimes p} \otimes \Omega_X^{\otimes q})$ is parallel with respect to ω.

The strategy consists in first taking a resolution $f : Y \to X$ and then applying Bochner formula to $f^*\sigma$ using a family of approximate Kähler-Einstein metrics ω_ε on Y converging to $f^*\omega$. The core of the matter is to deal with the error terms coming from the fact that $f^*\sigma$ may pick up poles along the exceptional divisor, and that ω_ε is not Ricci flat. This is quite technical, and will not be discussed here further.

An important consequence of the Bochner principle is that the Albanese map of a compact Kähler variety with canonical singularities and trivial first Chern class is surjective and locally trivial, generalizing results of Kawamata in the projective case [**Kaw85**]. More precisely,

***Theorem 8.2* ([CGGN22]).** — *Let X be a compact Kähler variety with canonical singularities and $c_1(X) = 0$, and let $\alpha : X \to A := \mathrm{Alb}(X)$ be the Albanese map of X. Then*

α is surjective with connected fibers, and there exists a finite étale cover $B \to A$ such that $X \times_A B \to B$ is isomorphic to the trivial fibration $F \times B \to B$.

In [**Gue16**] (cf [**GKP16b**] for similar results in the projective case), we proved that the tangent sheaf of X admits a polystable decomposition

$$T_X = \bigoplus \mathscr{E}_i$$

such that the induced subbundles $\mathscr{E}_i|_{X_{\mathrm{reg}}}$ are pairwise orthogonal and parallel with respect to ω. In particular, it is natural to ask if one can compute their holonomy $\mathrm{Hol}(\mathscr{E}_i, \omega)$, understood as the holonomy of the vector bundle $\mathscr{E}_i|_{X_{\mathrm{reg}}}$ with respect to $\omega|_{X_{\mathrm{reg}}}$. In [**GGK19, CGGN22**], we proved the following:

Theorem 8.3 **(Holonomy covers, [GGK19, CGGN22])**
Let (X, ω_0) be a compact Kähler variety with klt singularities and $c_1(X) = 0$, and let $\omega \in [\omega_0]$ be the singular Ricci flat metric. Then after replacing X by a finite quasi-étale cover, there exists a direct sum decomposition of the tangent sheaf of X,

$$T_X = \mathscr{F} \oplus \bigoplus_{k \in K} \mathscr{E}_k,$$

where the reflexive sheaves $\mathscr{F}$ and $\mathscr{E}_k$ satisfy the following:

(i) The sheaves $\mathscr{F}$ and $\mathscr{E}_k$ are foliations with trivial determinant.
(ii) The sheaf $\mathscr{F}|_{X_{\mathrm{reg}}}$ is flat. More precisely, it is given by a special unitary representation of $\pi_1(X_{\mathrm{reg}})$.
(iii) Each factor $\mathscr{E}_k\big|_{X_{\mathrm{reg}}}$ is parallel and has full holonomy group either $\mathrm{SU}(n_k)$ or $\mathrm{Sp}(n_k/2)$, with respect to the pullback of the singular Ricci-flat metric ω. Here, $n_k = \mathrm{rk}(\mathscr{E}_k)$. Moreover, $S^{[m]}\mathscr{E}_k$ is strongly stable with respect to any Kähler class, for any integer $m \geqslant 1$.

The proof of the above theorem goes roughly as follows. Decompose $T_X = \bigoplus \mathscr{E}_i$ into stable pieces. Up to taking a further quasi-étale cover, one can assume that the pieces are strongly stable (i.e. they remain stable after any further quasi-étale cover). Then, Bochner principle essentially guarantees that the component of the identity of the holonomy group, $G_i^\circ := \mathrm{Hol}^\circ(\mathscr{E}_i, \omega)$, is irreducible. By Berger-Simons classification, we have only three possibilities for G_k°, that is, the trivial group, $\mathrm{SU}(n_k)$ or $\mathrm{Sp}(n_k/2)$ where $n_k = \mathrm{rk}(\mathscr{E}_k)$. The first case correspond to flat factors, which we set aside. As for the other ones, we need to show that G_k/G_k° is finite, so that one can perform a finite quasi-étale cover and get the full holonomy to be what we want. In general, we only have a surjection

$$\pi_1(X_{\mathrm{reg}}) \to G_k/G_k^\circ$$

but the group on the left-hand side is not yet known to be finite (it is conjectured though, say for varieties with zero augmented irregularity). The key point is that since G_k° is normal in $G_k \subset \mathrm{U}(n_k)$, we know that G_k is included in the normalizer of $\mathrm{SU}(n_k)$ (resp. $\mathrm{Sp}(n_k/2)$) in $\mathrm{U}(n_k)$. In particular, one can see that

$$G_k/G_k^\circ \subset \mathrm{U}(1)$$

is abelian and we get a surjective map

$$H_1(X_{\mathrm{reg}}, \mathbb{Z}) \to G_k/G_k^\circ.$$

In [**GGK19, CGGN22**], we provide two independent ways to see that the right-hand side is finite. One appeals to Theorem 7.5 in the projective case while the other one consists in proving directly that $H_1(X_{\mathrm{reg}}, \mathbb{Z})$ is torsion by relying on a theorem of Deligne allowing us to compute the de Rham cohomology of X_{reg} via the coherent cohomology of a resolution of X.

9. The singular Beauville-Bogomolov decomposition theorem

9.1. The irreducible pieces in the decomposition. — Being able to detect whether a variety X is covered by a torus by looking at its tangent sheaf is a fundamental question in the theory; we will first give a criterion in terms of flatness of $T_{X_{\mathrm{reg}}}$ and in the next section we will give a purely numerical criterion, in the spirit of Theorem 7.6 and Theorem 7.9. The result below was obtained jointly with Claudon-Graf-Naumann:

Theorem 9.1 **([CGGN22]).** — *Let X be a compact Kähler variety with klt singularities such that $T_{X_{\mathrm{reg}}}$ is holomorphically flat. Then, there exists a finite quasi-étale cover $T \to X$ for some torus T.*

The two main tools involved in the proof are, as in the projective case, the solution of Zariski-Lipman conjecture for klt singularities as well as finiteness results for regional fundamental groups [**Xu14**]. As we already explained in § 7.2, that last result only holds for algebraic singularities. The new idea is to run a delicate induction process starting from isolated singularities which are always algebraic, and couple the induction with Zariski-Lipman.

As a consequence of Theorem 9.1, a compact Kähler variety X (of dimension at least two) with klt singularities, $c_1(X) = 0$ and T_X strongly stable cannot be flat. As a result, Theorem 8.3 combined with Bochner principle imply that X automatically falls into one of the following classes of varieties:

Definition 9.1 **(ICY and IHS varieties).** — Let X be a compact Kähler variety of dimension $n \geqslant 2$ with canonical singularities and $K_X \simeq \mathcal{O}_X$.

1. We call X *irreducible Calabi–Yau (ICY)* if $H^0(Y, \Omega_Y^{[p]}) = 0$ for all integers $0 < p < n$ and all quasi-étale covers $Y \to X$, in particular for X itself.
2. We call X *irreducible holomorphic symplectic (IHS)* if there exists a holomorphic symplectic two-form $\sigma \in H^0(X, \Omega_X^{[2]})$ such that for all quasi-étale covers $\gamma : Y \to X$, the exterior algebra of global reflexive differential forms is generated by $\gamma^{[*]}\sigma$.

The main result of this section is the following:

Theorem 9.2 **(Singular Beauville-Bogomolov decomposition Theorem, [Dru18, GGK19, HP19, BGL22])**

Let X be a compact Kähler variety with klt singularities and $c_1(X) = 0 \in H^2(X, \mathbb{R})$. Then, there exists a finite quasi-étale cover $\widetilde{X} \to X$ such that $\widetilde{X}$ decomposes as a Kähler variety as follows,

$$\widetilde{X} = T \times \prod_i Y_i \times \prod_j Z_j,$$

where T is a complex torus, and where the Y_i (resp. Z_j) are irreducible Calabi-Yau varieties (resp. irreducible holomorphic symplectic varieties).

9.2. The projective case. — Given what we explained above, proving the theorem amounts to showing that the irreducible factor appearing in the decomposition

$$T_X = \mathscr{F} \oplus \bigoplus_{k\in K} \mathscr{E}_k$$

from Theorem 8.3 come from the tangent sheaf of a factor of a splitting of $X = T \times \prod_k X_k$ itself, maybe up to a cover. Indeed, T would then have flat tangent sheaf, hence it would be covered by a torus, while X_k would have holonomy $\mathrm{SU}(n_k)$ or $\mathrm{Sp}(n_k/2)$ and it would be an ICY or IHS by the Bochner principle.

The proof of the splitting is due to Druel [**Dru18**] and Höring-Peternell [**HP19**]. More precisely, Druel first showed that the flat factor $\mathscr{F}$ does indeed come from a torus splitting from X after some quasi-étale cover. Moreover, he proved that it would then be sufficient to show that the foliations $\mathscr{E}_k$ are algebraically integrable, that is, that their leaves (defined as immersed submanifolds of X_{reg}) are actually algebraic varieties, or equivalently that they are open in their Zariski closure in X.

Later, Höring-Peternell proved that the strong stability of the reflexive symmetric powers $S^{[m]}\mathscr{E}_k$ implies that $\mathscr{E}_k^*$ was *not* pseudoeffective, i.e. there exists an ample divisor H as well as a number $c > 0$ such that for all i, j satisfying $i > cj$, we have

$$H^0(X, S^{[i]}\mathscr{E}_k \otimes \mathcal{O}_X(jH)) = 0.$$

In turn, Druel had generalized a theorem of Bost [**Bos01**] (see also [**CP19**]) asserting that the above condition was sufficient to guarantee the algebraic integrability of the leaves of $\mathscr{E}_k$.

9.3. The Kähler case. — In this paragraph, we will briefly survey the main result of [**BGL22**] where we extend the decomposition theorem from the projective case to the Kähler case. Although the proof is overall quite lengthy and technical, the strategy is pretty clear and the main idea or starting point is to approximate X by projective varieties and use the decomposition theorem in the algebraic case. The first step is achieved in the following

Theorem 9.3. — *Any X as in Theorem 9.2 admits a strong locally trivial algebraic approximation: there is a locally trivial family $\mathcal{X} \to S$ over a smooth base S specializing to X over $s_0 \in S$ such that points $s \in S$ for which X_s is projective are analytically dense near s_0.* □

Let us emphasize that we are not claiming that the Bogomolov–Tian–Todorov theorem holds in this context—that is, that locally trivial deformations of numerically K-trivial X as in the theorem are always unobstructed (which would be sufficient to prove Theorem 9.3, see [**GS21a**, Theorem 1.2]). However, we can find a locally trivial deformation that is well-enough behaved so that the base space is smooth, and general enough so that one can find nearby projective fibres.

The main difficulty in the proof of Theorem 9.3 is therefore to produce sufficiently many unobstructed deformations. Recall that by Kodaira's criterion, a compact Kähler manifold (or compact Kähler space with rational singularities) with no nonzero holomorphic 2-forms is automatically projective, so the existence of 2-forms is the only obstruction to projectivity. The results of [**Gue16**] extending the polystability of T_X to the Kähler category provide a splitting of T_X into foliations and the symplectic foliations among the factors account for all of the 2-forms on X. More precisely, one can decompose

$$T_X = E \oplus F = \left(\bigoplus_{i=1}^{r} E_i \right) \oplus F,$$

where for all $i = 1, \ldots, r$, E_i is stable and there exists an index j and an isomorphism $E_i \simeq E_j^*$, and $H^0(X, \Lambda^2 F^*) = 0$. The isomorphism $E_i \simeq E_j^*$ can be realized as the contraction map by a reflexive 2-form, which can be assumed to be non-degenerate on E_i if $j = i$. Heuristically (but a posteriori true on a quasi-étale cover), E corresponds to the torus and irreducible holomorphic symplectic factors while F corresponds to the irreducible Calabi-Yau factors. Note that E is not necessarily symplectic, as the example of a 3-torus shows (there, $F = 0$ but there is no symplectic form).

It is therefore natural to try to deform to the symplectic directions $H^1(X, E) \subset H^1(X, T_X)$ and we show that locally trivial deformations obtained by exponentiating the symplectic foliations of this splitting are always unobstructed. Moreover, one gets surjectivity of the map obtained by contracting a Kähler class $H^1(X, E) \to H^2(X, \mathcal{O}_X)$ almost automatically, hence one can apply the singular version of Green-Voisin criterion proved by Graf-Schwald [**GS21a**] in order to find nearby projective fibers.

With Theorem 9.3 in hand, the proof of Theorem 9.2 proceeds as follows. We first produce a locally trivial deformation $\pi : \mathcal{X} \to \Delta$ of X over the disk for which projective fibers are analytically dense. By cycle-theoretic arguments and Theorem 9.2 in the projective case, replacing X by a quasi-étale cover it suffices to assume there is a splitting $\mathcal{X}^* = \mathcal{Y}^* \times_{\Delta^*} \mathcal{Z}^*$ of the family $\mathcal{X}^* := \mathcal{X}|_{\Delta^*}$ over the punctured disk, and we must show that the splitting extends over the puncture.

One first observes that local triviality of the family $\pi : \mathcal{X} \to \Delta$ implies the Künneth decomposition of $R^k \pi_* \mathbb{Q}_{\mathcal{X}^*}$ extends, in fact as a decomposition of the variation of Hodge structures. By K-triviality, the factors of the splitting of the tangent bundle $T_{\mathcal{X}^*/\Delta^*}$ are cut out by differential forms and extend, so we have a splitting $T_{\mathcal{X}/\Delta} \cong A \oplus B$. The leaves of the splitting of the family over Δ^* have well-defined limits in the special fiber which are therefore tangent to the factors of the limit splitting $T_X = A_0 \oplus B_0$ on the regular locus X^{reg}.

It remains to show that the limit leaves define a product structure in the singular locus X^{sing}. There are essentially two types of phenomena that could go wrong:

(*i*) the limit leaves could acquire new components in X^{sing}, or
(*ii*) limit leaves in the two directions could have positive-dimensional intersections in X^{sing}.

To rule these out, we show that the splitting of the Ricci-flat metric $\omega_t = \omega_{A_t} + \omega_{B_t}$ of X_t for $t \in \Delta^*$ induced by the splitting of the family extends over the puncture to a decomposition $\omega_0 = \omega_{A_0} + \omega_{B_0}$ of the Ricci-flat metric on $X_0 = X$ into closed positive currents with bounded potentials. This is the key technical part of the proof of Theorem 9.2

and the latter condition is critical: it implies that these currents can be restricted to cycles in the singular locus and that one can compute intersection numbers with these currents. The fact that the decomposition is compatible with the limit Künneth decomposition and the semipositivity of the factors together imply neither pathology arises.

Let us give a simplified version of the argument to illustrate how the proof of (*ii*) goes. Let $F \subset X_0$ (resp. $G \subset X_0$) be a cycle obtained as limit of cycles of the form $Y_t \times \{z_t\}$ (resp. $\{y_t\} \times Z_t$) for some point $z_t \in Z_t$ (resp. $y_t \in Y_t$). For simplicity, assume that $F, G \not\subset X_0^{\mathrm{sing}}$. Note that the cycles $Y_t \times \{z_t\}$ will always converge (after extracting a subsequence $t_i \to 0$) by properness of the irreducible components of the relative Douady space; this in turn relies on Bishop's theorem and the fact that the fibers of π are Kähler.

One wants to show that $F \cap G$ is finite. On the regular locus of X_0, the semipositive form ω_{A_0} satisfies $\ker(\omega_{A_0}) = B_0$. Since G is a limit of leaves G_t tangent to B_t, G is tangent to B_0 and

$$\omega_{A_0}|_G \equiv 0 \quad \text{on} \quad X_0^{\mathrm{reg}}.$$

Similarly, $\omega_{B_0}|_F \equiv 0$ on X_0^{reg}. The decomposition $\omega_0 = \omega_{A_0} + \omega_{B_0}$ implies that

$$\omega_0|_{F\cap G} \equiv 0 \quad \text{on} \quad X_0^{\mathrm{reg}}.$$

Now, the crucial information that ω_0 has bounded potentials implies that $\omega_0|_{F\cap G}$ is well-defined and vanishes *everywhere*. That information also allows us to say that the volume of $F \cap G$ with respect to the Kähler class $[\omega_0]$ is nothing but

$$\int_{F\cap G} \omega_0^{\dim F\cap G} = 0,$$

hence $F \cap G$ is finite.

10. Numerical characterization of complex torus quotients

In this section paragraph, we explain the following (partial) generalization of Theorem 7.9 to the Kähler case that we obtained jointly with Claudon and Graf.

Theorem 10.1 **([CGG22]).** — *Let X be a compact Kähler variety of dimension n with klt singularities. Assume that X is smooth in codimension two. The following assertions are equivalent*

(*i*) *There exists a torus T and some finite group $G \subset \mathrm{Aut}(T)$ acting freely in codimension two such that*

$$X \simeq A/G.$$

(*ii*) *We have $c_1(X) = 0 \in H^2(X, \mathbb{R})$, and there exists a Kähler class $\alpha \in H^2(X, \mathbb{R})$ such that*

$$\hat{c}_2(X) \cdot \alpha^{n-2} = 0.$$

Let us now explain the proof of the above theorem. It relies on three tools:

- A singular Bogomolov-Gieseker inequality.
- The singular Beauville-Bogomolov decomposition theorem, i.e. Theorem 9.2.
- A formula computing c_2 for irreducible holomorphic symplectic varieties.

Step 1. *A singular Bogomolov-Gieseker inequality*

Proposition 10.1. — *Let X be a normal compact Kähler variety of dimension n together with a Kähler class $\alpha \in H^2(X, \mathbb{R})$. Assume that X is smooth in codimension two. Furthermore, let $\mathscr{E}$ be a rank r reflexive coherent sheaf on X which is slope stable with respect to α.*

(1) *The discriminant $\Delta(\mathscr{E}) := 2r\, c_2(\mathscr{E}) - (r-1)\, c_1^2(\mathscr{E})$ satisfies the inequality*

$$\Delta(\mathscr{E}) \cdot \alpha^{n-2} \geqslant 0.$$

(2) *If equality holds, then we have*

$$\Delta(\mathscr{E}) \cdot \beta^{n-2} = 0$$

for any *Kähler class $\beta \in H^2(X, \mathbb{R})$.*

The inequality part is almost straightforward: take a log resolution $f : Y \to X$, then $f^{[*]}\mathscr{E}$ is $f^*\alpha$-stable, hence it remains stable with respect to $f^*\alpha + \varepsilon[\omega_Y]$ for some Kähler form ω_Y and $0 \leqslant \varepsilon \ll 1$. Then one applies the usual Bogomolov-Gieseker inequality, and pass to the limit when $\varepsilon \to 0$. The second half of the statement is a bit less standard but it essentially relies on the Kobayashi-Hitchin correspondence providing a family of Hermite-Einstein metrics h_ε for $f^{[*]}\mathscr{E}$.

Step 2. *Using the decomposition theorem*

Thanks to Theorem 9.2, one can pass to a quasi-étale cover such that X becomes isomorphic to a product

$$X \simeq T \times \prod_{i \in I} Y_i \times \prod_{j \in J} Z_j$$

where Y_i are ICY and Z_j are IHS. The crucial observation is that α decomposes as a sum

$$\alpha = \mathrm{pr}_T^* \alpha_T + \sum \mathrm{pr}_{Y_i}^* \alpha_{Y_i} + \sum \mathrm{pr}_{Z_i}^* \alpha_{Z_j}$$

where $\alpha_T, \alpha_{Y_i}, \alpha_{Z_j}$ are Kähler classes on T (resp. Y_i, Z_j). This is because of Künneth formula and the vanishing $H^1(Y_i, \mathbb{C}) = H^1(Z_j, \mathbb{C}) = 0$ thanks to the absence of 1-forms on these varieties with rational singularities. Proposition 10.1 then implies

$$c_2(Y_i) \cdot \alpha_{Y_i}^{\dim Y_i - 2} = c_2(Z_j) \cdot \alpha_{Z_j}^{\dim Z_j - 2} = 0,$$

and we are left to showing that this is absurd unless $I = J = \emptyset$.

Step 3. *Strict positivity of c_2 for ICY and IHS varieties.*

• *Case 1.*

Let X be an ICY variety of dimension n such that there exists a Kähler class α satisfying $c_2(X) \cdot \alpha^{n-2} = 0$.

Since $H^0(X, \Omega_X^{[2]}) \simeq H^2(X, \mathscr{O}_X) = 0$, X is projective. Moreover, the equality case of Proposition 10.1 implies that one can find an ample divisor H such that $c_2(X) \cdot H^{n-2} = 0$. By Theorem 7.9, this implies that X is covered by an abelian variety, a contradiction with the definition of ICY variety.

• *Case 2.*

Let X be an ICY variety of dimension n such that there exists a Kähler class α satisfying $c_2(X) \cdot \alpha^{n-2} = 0$.

We claim that there exists a non-negative constant $C \geqslant 0$ such that

$$\forall a \in H^2(X, \mathbb{C}), \quad c_2(X) \cdot a^{n-2} = C \cdot q_X(a)^{\frac{n}{2}-1}. \tag{10.1}$$

where q_X is the Beauville-Bogomolov-Fujiki quadratic form on $H^2(X, \mathbb{C})$. The formula (10.1) can be derived just like in the smooth case from the existence of a universal locally trivial deformation space along with the Torelli theorem [**BL22**]. Its non-negativity follows e.g. from Proposition 10.1.

Since $q_X(\alpha)^{\frac{n}{2}} = \alpha^n > 0$ up to a positive universal constant, one infers from the vanishing of $c_2(X) \cdot \alpha^{n-2}$ and (10.1) that $C = 0$. In particular, one gets

$$\forall a \in H^2(X, \mathbb{C}), \quad c_2(X) \cdot a^{n-2} = 0.$$

The key point is that this property is now deformation invariant (whereas the parallel transport of a $(1,1)$ class has no type in general). Take a locally trivial deformation $\pi : \mathcal{X} \to \mathbb{D}$ such that X_t is projective some some $t \in \mathbb{D}$, whose existence is due to [**BL22**]. Then for any (ample) divisor H_t on X_t, we have $c_2(X_t) \cdot H_t^{n-2} = 0$, hence X_t is covered by an abelian variety thanks to Theorem 7.9. Since π is locally trivial, the inclusion $X_t^{\mathrm{reg}} \hookrightarrow \mathcal{X}^{\mathrm{reg}}$ yields an isomorphism $\pi_1(X_t^{\mathrm{reg}}) \to \pi_1(\mathcal{X}^{\mathrm{reg}})$, hence one can find a quasi-étale cover $\mathcal{Y} \to \mathcal{X}$ such that Y_t is abelian. In particular, $\mathcal{Y} \to \mathbb{D}$ is smooth, and $h^0(Y_0, \Omega^1_{Y_0}) \neq 0$. Since $Y_0 \to X_0$ is quasi-étale, this contradicts the fact that X_0 is an IHS variety.

11. Families of log Calabi-Yau manifolds

Definition 11.1. — A log Calabi-Yau manifold consists in a pair (X, B) where

- X is a compact Kähler manifold.
- $B = \sum b_i B_i$ is a divisor with snc supports and coefficients $b_i \in (0,1)$.
- The class $c_1(K_X + B) \in H^2(X, \mathbb{R})$ vanishes.

Associated to a log Calabi-Yau manifold (X, B) are Kähler Ricci flat cones metrics, as constructed by [**Bre13, CGP13, GP16, JMR16**]. These are Kähler Ricci-flat metrics ω on $X \setminus \mathrm{Supp}(B)$ that are locally quasi-isometric to the model cone metric

$$\omega_{\mathrm{mod}} := \sum_{k=1}^{p} \frac{i dz_k \wedge d\bar{z}_k}{|z_k|^{2b_k}} + \sum_{k \geqslant p+1} i dz_k \wedge d\bar{z}_k$$

on each coordinate chart $(U, (z_i))$ where $U \cap B_k = (z_k = 0)$ for $1 \leqslant k \leqslant p$, up to relabelling the B_i's. Such a metric is incomplete, has finite volume and automatically extends to a closed, positive $(1,1)$-current on X, satisfying the equation

$$\mathrm{Ric}\, \omega = [B] \tag{11.1}$$

understood in the sense of currents.

In this section, we would like to explain how these Kähler Ricci flat cone metrics help understand the geometry of (X, B), especially in a context of families of such manifolds.

11.1. The relative Ricci flat metric is not semipositive. — In this paragraph, we explain a negative result in the case $B = 0$ that will motivate the result in the next paragraph when B may not be zero anymore.

The context is as follows. Let $p : X \to Y$ be a holomorphic fibration between projective manifolds of relative dimension $n \geqslant 1$. Let X° be the set of regular values, and let $X^\circ := p^{-1}(Y^\circ)$. We assume that for $y \in Y^\circ$, $c_1(K_{X_y}) = 0$, where $X_y := p^{-1}(y)$. Let L be a pseudoeffective, p-ample $\mathbb{Q}$-line bundle on X. One can write $L = H + p^*M$ for some ample line bundle H on X and for some line bundle M on Y. In particular, one can find a smooth $(1,1)$-form $\omega \in c_1(L)$ on X such that for any $y \in Y^\circ$, $\omega_y := \omega|_{X_y}$ is a Kähler form on X_y.
By Yau's theorem, there exists for any $y \in Y^\circ$ a unique function $\varphi_y \in \mathcal{C}^\infty(X_y)$ such that:

(*i*) $\theta_y := \omega_y + dd^c\varphi_y$ is a Kähler form
(*ii*) $\int_{X_y} \varphi_y \omega_y^n = 0$
(*iii*) $\mathrm{Ric}\,\theta_y = -dd^c \log \omega_y^n = 0$

Moreover, one can use the implicit function theorem to check that the dependence of φ_y in y is smooth, so that the form $\theta := \omega + dd^c\varphi$ is a well-defined smooth $(1,1)$-form on X° which is relatively Kähler. A folklore conjecture asserted that the form θ is semipositive on X, say when L is globally ample. This turns out to be wrong:

Theorem 11.1 ([**CGP19**]). — *There exists a projective fibration $p : X \to Y$ as in the setting above and an ample line bundle L on X such that the relative Ricci-flat metric θ on X° associated with L is not semipositive.*

Remark 11.2. — The counter-example is actually pretty explicit: X is a $K3$ surface and p is an elliptic fibration onto $Y = \mathbb{P}^1$.

Idea of the proof of Theorem 11.1. — Let $p : X \to \mathbb{P}^1$ be a non-isotrivial elliptic fibration admitting another transversal elliptic fibration $q : X \to P^1$. First, a relatively harmless (but crucial) reduction allows us assume that $L = q^*\mathcal{O}_{\mathbb{P}^1}(1)$ is semiample with $L^2 = 0$.

If θ were semipositive on X°, then one could extend it to a positive current in $c_1(L)$ (cf appendix of [**CGP19**] by Valentino Tosatti, or [**DGG20**] for a more general argument). Necessarily, one would have $\theta^2 \equiv 0$ on X°, otherwise L would be big. One some Zariski open set of X, the foliation $\ker(\theta)$ coincides with that of $q : X \to \mathbb{P}^1$. As the lift V of $\frac{\partial}{\partial t}$ with respect to θ belongs to $\ker(\theta)$, a short computation shows that V is actually holomorphic, hence its identifies the smooth fibers of p biholomorphically, a contradiction with the non-isotriviality of p. □

11.2. A metric criterion for local triviality. — In the recent years, metric criteria to prove the local triviality of a given fibration $p : X \to Y$ have been at the heart of breakthroughs, especially in relation with the structure of projective varieties X with $-K_X$ nef, using canonical fibrations such that the Albanese map or the MRC fibration, cf. [**Cao19, CH19, CCM21, MW21**] and references therein. A common feature in the results there is the existence of a flat hermitian bundle on the base Y, usually obtained as direct image of a bundle upstairs, having to do with the pluricanonical bundles. The main result of this paragraph, Theorem 11.3, provides a similar triviality criterion in the context of families of log Calabi-Yau manifolds.

- **Geometric setting.**

In this paragraph, we will be working under the following geometric assumptions. Let $p : (X, B) \to Y$ be a proper, holomorphic fibration between two Kähler manifolds, where $B = \sum b_i B_i$ is an effective $\mathbb{Q}$-divisor on X whose coefficients $b_i \in (0, 1)$ are smaller then one. We assume that there exists $Y^\circ \subset Y$ contained in the smooth locus of p such that $B|_{X_y}$ has snc support and set $X^\circ := p^{-1}(Y^\circ)$. The fibers of p are assumed to satisfy

$$c_1(K_{X_y} + B|_{X_y}) = 0 \quad \text{for any } y \in Y^\circ. \tag{11.2}$$

- **Direct image of the relative pluricanonical bundle.**

For any integer $m \geqslant 1$, one defines

$$\mathcal{F}_m := p_\star\big(m(K_{X/Y} + B)\big)^{\star\star}$$

which is a reflexive sheaf on Y, enjoying many properties. One of them is to admit a canonical metric h, called Narasimhan-Simha metric, enjoying many special curvature properties as observed e.g. [**BP08, PT18**].

We take advantage of our simple geometric situation (i.e. the condition (11.2)) to offer a more concise definition of h, since $\mathcal{F}_m$ can be showed to be a line bundle as follows. Thanks to the log abundance in the Kähler setting, cf. [**CGP19**, Corollary 1.18] and references therein, we know that $K_{X_y} + B_y$ is $\mathbb{Q}$-effective. Combining this with Ohsawa-Takegoshi extension theorem in its Kähler version, cf. [**Cao17**], one can improve (11.2) by obtaining an integer $m \geqslant 1$ such that

$$m(K_{X_y} + B|_{X_y}) \simeq \mathcal{O}_{X_y} \quad \text{for any } y \in Y^\circ. \tag{11.3}$$

Given a point $y \in Y^\circ$, there exists a coordinate ball $U \subset Y^\circ$ containing y and a nowhere vanishing holomorphic section

$$\Omega \in H^0\big(X_U, m(K_{X/Y} + B)|_{X_U}\big) \tag{11.4}$$

where $X_U := p^{-1}(U)$. If f_B is a local multivalued holomorphic function cutting out the $\mathbb{Q}$-divisor B, then the form $\frac{(\Omega_y \wedge \overline{\Omega}_y)^{\frac{1}{m}}}{|f_B|^2}$ induces a volume element on the fibers of p over U, and one sets

$$V_y := \int_{X_y} \frac{(\Omega_y \wedge \overline{\Omega}_y)^{\frac{1}{m}}}{|f_B|^2}.$$

Now, let $\sigma \in H^0(U, \mathcal{F}_m|_U)$ be a local holomorphic section of the line bundle $\mathcal{F}_m$ defined over a small coordinate set $U \subset Y^\circ$. The expression

$$\|\sigma\|_y^2 := V_y^{m-1} \int_{X_y} \frac{|\sigma|^2}{|\Omega_y|^{2\frac{m-1}{m}}} e^{-\phi_B} \tag{11.5}$$

defines a metric h on $\mathcal{F}_m|_{Y^\circ}$. This metric extends across the singularities of the map p, and it has semi-positive curvature current, see [**BP08, PT18**].

- **The relative Kähler Ricci-flat cone metric.**

If we fix a reference Kähler form ω on X, then we can construct a fiberwise Ricci-flat conic Kähler θ_y metric, i.e. a solution of the equation

$$\begin{cases} \operatorname{Ric}\theta_y = [B_y] \\ \theta_y \in [\omega_y] \end{cases}.$$

There exists a unique function $\varphi \in L^1_{\mathrm{loc}}(X^\circ)$ such that

$$\begin{cases} \theta_y = \omega_y + dd^c\varphi|_{X_y} \\ \int_{X_y} \varphi\, \omega_y^n = 0 \end{cases}.$$

The closed $(1,1)$-current $\theta^\circ_{\mathrm{KE}} := \omega + dd^c\varphi$ on X° is called *relative Kähler Ricci-flat cone metric* in $[\omega]$.

As we saw in Theorem 11.1, the current $\theta^\circ_{\mathrm{KE}}$ is not positive in general, which marks an important difference with the case of Kähler fiber spaces whose generic fiber is of general type, cf Theorems 1.3-5.2.

Theorem 11.3. — *Let $p : (X, B) \to Y$ be a map as above, and let ω be a fixed Kähler metric on X. Assume that the following conditions are satisfied.*

(*i*) *For $y \in Y^\circ$, the $\mathbb{Q}$-line bundle $K_{X_y} + B_y$ is numerically trivial.*
(*ii*) *For some m large enough, the line bundle $p_*(m(K_{X^\circ/Y^\circ} + B))$ is Hermitian flat with respect to the Narasimhan-Simha metric h on Y°, cf* (11.5).

Then, we have the following.

(†) *The relative Kähler Ricci-flat cone metric $\theta^\circ_{\mathrm{KE}}$ is positive and it extends canonically to a closed positive current $\theta_{\mathrm{KE}} \in \{\omega\}$ on X.*
(‡) *The fibration $(X, B) \to Y$ is locally trivial over Y°. Moreover, if p is smooth in codimension one and $\operatorname{codim}_X(B \smallsetminus X^\circ) > 1$, then p is locally trivial over the whole Y.*

The last statement means that for every $y \in Y^\circ$, there exists a neighborhood $U \subset Y^\circ$ of y such that

$$(p^{-1}(U), B) \simeq (X_y, B|_{X_y}) \times U.$$

Idea of the proof. — The key object in the proof is $\theta^\circ_{\mathrm{KE}}$. First, we show that it is semi-positive as writing it as a limit of negatively curved KE cone metrics using [**Gue20a**]. Of course, the flatness of the direct image is crucial here.

Next, the idea is to use the PDE that the geodesic curvature $c(\theta)$ of $\theta := \theta^\circ_{\mathrm{KE}}$ over a one-dimensional disk $\Delta \subset Y$ is solving on a given fiber X_y:

$$-\Delta''_\theta\,(c(\theta)) = |\bar\partial v_\theta|^2 - \Theta(K_{\mathcal{X}/\Delta}, (\theta^n)^{-1})(v_\theta, v_\theta)$$

where v_θ is the lift of $\frac{\partial}{\partial t}$ with respect to θ in order to produce a holomorphic vector field $v = v_\theta$ which will identify the fibers X_y and the divisors B_y. The holomorphicity of v should be derived from integrating the formula above on X_y and using the flatness of the direct image so that the last term in the RHS integrates to zero. Unfortunately, one cannot work with the singular metric θ directly and one needs to regularize it, leading to lengthy computations to ensure that the strategy can indeed be carried out. □

11.3. Applications to families of log Calabi-Yau manifolds. — Theorem 11.3 above has many geometric applications, like for instance a Kähler version of a theorem of Ambro [**Amb05**] and a log version of Păun's result [**Pău17**] on the surjectivity of the Albanese map of a compact Kähler manifold with $-K_X$ nef.

Corollary 11.4. — *Let $p: X \to Y$ be a fibration between two compact Kähler manifolds. Let B be a $\mathbb{Q}$-effective klt divisor on X with snc support.*

- *If $-(K_X + B)$ is nef, then $-K_Y$ is pseudo-effective. Moreover, the Albanese map of X is surjective.*
- *Moreover, if $c_1(K_X + B) = 0$ and $c_1(Y) = 0$, then p is locally trivial, that is, for every $y \in Y$, there exists a neighborhood $U \subset Y$ of y such that*
$$(p^{-1}(U), B) \simeq (X_y, B|_{X_y}) \times U.$$
In particular, if $c_1(K_X + B) = 0$, the Albanese map $p: X \to \mathrm{Alb}(X)$ is locally trivial.

Another striking consequence is the following positivity property of direct images of pluri-log canonical bundles. It can be seen as a logarithmic version of Viehweg's $Q_{n,m}$-conjecture for families of log Calabi-Yau manifolds, cf [**Vie83**].

Corollary 11.5. — *Let $p: (X, B) \to Y$ be a fibration between two compact Kähler manifolds such that $c_1\left(K_{X_y} + B|_{X_y}\right) = 0$ for a generic $y \in Y$. Assume moreover that the logarithmic Kodaira-Spencer map*

$$T_Y \to \mathcal{R}^1 p_\star\left(T_{X/Y}(-\log B)\right) \tag{11.6}$$

is generically injective. Then the bundle $p_\star\left(m(K_{X/Y} + B)\right)^{\star\star}$ is big.

We remark that, based on Corollary 11.5 and some deep tools, Y. Deng [**Den19**] proved recently the hyperbolicity of bases of maximally variational smooth families of log Calabi-Yau pairs.

PART III. POSITIVE CURVATURE

12. A decomposition theorem for Kähler-Einstein $\mathbb{Q}$-Fano varieties

12.1. Kähler-Einstein metrics on Fano varieties. — Let (X, ω) be a Fano Kähler-Einstein manifold, i.e. X is a projective manifold with $-K_X$ ample and admitting a Kähler metric ω solving $\mathrm{Ric}\,\omega = \omega$. It follows from the (easy direction of the) Kobayashi-Hitchin correspondence that the tangent bundle of X splits as a direct sum of parallel subbundles

$$T_X = \bigoplus_{i \in I} F_i \tag{12.1}$$

such that F_i is stable with respect to $-K_X$. Since X is simply connected, de Rham's splitting theorem asserts that one can integrate the foliations arising in decomposition (12.1) and obtain an isometric splitting

$$(X, \omega) \simeq \prod_{i \in I} (X_i, \omega_i) \tag{12.2}$$

into Kähler-Einstein Fano manifolds which is compatible with (12.1).

Over the last few decades, a lot of attention has been drawn to projective varieties with mild singularities, in relation to the progress of the Minimal Model Program (MMP). In that context, the notion of $\mathbb{Q}$-Fano variety has emerged and played a central role in birational geometry. Recall that we say that a projective variety X is a $\mathbb{Q}$-Fano variety if X has klt singularities and $-K_X$ is an ample $\mathbb{Q}$-line bundle.

On the analytic side, singular Kähler-Einstein metrics with positive curvature on a $\mathbb{Q}$-Fano variety X have been introduced and studied in [**BBE$^+$19**]:

Definition 12.1. — Let X be a $\mathbb{Q}$-Fano variety. A Kähler-Einstein metric is a closed, positive current $\omega \in c_1(X)$ with bounded potentials, which is smooth on X_{reg} and satisfies

$$\mathrm{Ric}\,\omega = \omega$$

on that open set.

Note that the condition on the boundedness of the potentials can be replaced by the volume condition: $\int_{X_{\mathrm{reg}}} \omega^n = c_1(X)^n$. Singular Kähler-Einstein metrics with positive curvature have played a major role in the resolution of the Yau-Tian-Donaldson conjecture, cf [**CDS15a, CDS15b, CDS15c**], as they naturally appear as Gromov-Hausdorff limits of smooth Fano manifolds along some continuity method.

12.2. Polystability of the canonical extension and applications. — A straightforward consequence of Myers theorem is that $(X_{\mathrm{reg}}, \omega)$ is geodesically incomplete unless X is smooth. This prevents the use of most useful results in differential geometry (like the de Rham's splitting theorem mentionned above) to analyze their behavior. However, these objects are well-suited to study (poly)-stability properties of the tangent sheaf as it was observed by [**Gue16**], relying on earlier results by [**Eno88**]. In the positive curvature case, new difficulties arise since one cannot regularize the singular Kähler-Einstein metrics with an equally good control on the Ricci curvature.

Let us recall the definition of the so-called canonical extension of T_X for a $\mathbb{Q}$-Fano variety X. Let $\mathscr{F}$ be a coherent sheaf on X sitting in the exact sequence below

$$0 \longrightarrow \Omega_X^{[1]} \longrightarrow \mathscr{F} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{12.3}$$

The sheaf $\mathscr{F}$ is automatically torsion-free and it is locally free on X_{reg}.

From now on, one assumes that the extension class of $\mathscr{F}$ is the image of $c_1(X)$ in $H^1(X, \Omega_X^1)$ under the canonical map

$$\mathrm{Pic}(X) \otimes \mathbb{Q} \simeq H^1(X, \mathcal{O}_X^*) \otimes \mathbb{Q} \to H^1(X, \Omega_X^1) \to H^1(X, \Omega_X^{[1]}).$$

This is legitimate since K_X is $\mathbb{Q}$-Cartier.

***Definition 12.2* (Canonical extension of T_X). —** The dual $\mathscr{E} := \mathscr{F}^*$ of the sheaf $\mathscr{F}$ sitting in the exact sequence (12.3) with extension class $c_1(X)$ is called the canonical extension of T_X by $\mathcal{O}_X$.

The exact sequence (12.3) is locally splittable since for any affine $U \subset X$, one has $h^1(U, \Omega_U^{[1]}) = 0$. In particular, when one dualizes (12.3), one see that the canonical extension of T_X by $\mathcal{O}_X$ sits in the short exact sequence below

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathscr{E} \longrightarrow T_X \longrightarrow 0. \tag{12.4}$$

In [**Tia92**, Theorem 0.1], Tian proved that if X is a Fano manifold admitting a Kähler-Einstein metric ω, then the canonical extension E of T_X by $\mathcal{O}_X$ admits an Hermite-Einstein metric with respect to ω; in particular, it is polystable with respect to $-K_X$. Of course, this property is stronger than the polystability of T_X.

With Stéphane Druel and Mihai Păun, we generalized this statement to the singular setting:

***Theorem 12.1* ([DGP20]). —** *Let X be a $\mathbb{Q}$-Fano variety admitting a Kähler-Einstein metric. Then the canonical extension $\mathscr{E}$ of T_X by $\mathcal{O}_X$ is polystable with respect to $c_1(X)$.*

Strategy of proof of Theorem 12.1.

The first step is to focus on T_X and prove that it is the direct sum of stable subsheaves that are parallel with respect to the Kähler-Einstein metric ω on X_{reg}. This is achieved by computing slopes of subsheaves using the metric induced by the Kähler-Einstein metric and using Griffiths' well-known formula for the curvature of a subbundle. However, the presence of singularities (for X and ω) makes it hard to carry out the analysis directly on X. One has to work on a resolution using approximate Kähler-Einstein metrics as in [**Gue16**]. Yet an additional error term appears in the Fano case, requiring to introduce some new ideas to deal with it.

The next step deals with the canonical extension $\mathscr{E}$. It relies largely on the computations carried out to prove the polystability of T_X, but on top of those, several new ideas are needed to overcome the presence of singularities. First, one needs reduce the statement to one on a resolution in order to use analytic methods. Then we use again the technique of working with approximate Kähler-Einstein metrics, but in the current context this has the effect of modifying the canonical extension as well. As a result, we cannot evaluate directly the slope of a subsheaf of the canonical extension corresponding to the initial Kähler-Einstein metric. Dealing with this difficulty is our main contribution in this framework. The rest of the proof uses a combination of the original idea of Tian and the slope computations for subsheaves of T_X.

In [**GKP22**], Greb-Kebekus-Peternell proved the following uniformization theorem:

Theorem 12.2 **([GKP22]).** — *Let X be a $\mathbb{Q}$-Fano variety such that the canonical extension of T_X is semistable with respect to to K_X. Then, the following assertions are equivalent:*

(i) There exists a finite group $G \subset \mathrm{PGL}(n+1, \mathbb{C})$ acting freely in codimension one such that
$$X \simeq \mathbb{P}^n/G.$$
(ii) The Miyaoka-Yau discriminant vanishes:
$$(2(n+1)\widehat{c}_2(X) - n\widehat{c}_1^2(X)) \cdot c_1(X)^{n-2} = 0.$$

Combining Theorem 12.1 and Theorem 12.2, we get the following singular version of the usual characterization of $\mathbb{P}^n$ as the unique Kähler-Einstein Fano manifold with vanishing Miyaoka-Yau discriminant:

Corollary 12.3. — *Let X be a Kähler-Einstein $\mathbb{Q}$-Fano variety. Then, X is isomorphic to $\mathbb{P}^n/G$ for some finite group $G \subset \mathrm{PGL}(n+1, \mathbb{C})$ acting freely in codimension one if and only if the Miyaoka-Yau discriminant of X vanishes:*
$$(2(n+1)\widehat{c}_2(X) - n\widehat{c}_1^2(X)) \cdot c_1(X)^{n-2} = 0.$$

12.3. The decomposition theorem. — At this point, one would like to rely on the polystability of T_X and more precisely its decomposition as direct sum of foliations $T_X = \oplus \mathscr{E}_i$ to integrate these into a splitting of X, similarly to the smooth case (12.2), and inspired by the Ricci-flat case [**GGK19, Dru18, HP19**]. In the Fano case, the algebraic integrability of $\mathscr{E}_i$ is a direct consequence of [**BM16**] since T_X is semistable of *positive* slope. However, new difficulties also arise since the singularities are klt rather than canonical and Gorenstein. Our main result in [**DGP20**] is obtained by combining the polystability statement above with a general splitting result for algebraically integrable foliations holding for projective varieties of klt type:

Theorem 12.4 **([DGP20]).** — *Let X be a $\mathbb{Q}$-Fano variety admitting a Kähler-Einstein metric ω. Then T_X is polystable with respect to $c_1(X)$. Moreover, there exists a quasi-étale cover $f\colon Y \to X$ such that $(Y, f^*\omega)$ decomposes isometrically as a product*
$$(Y, f^*\omega) \simeq \prod_{i\in I} (Y_i, \omega_i),$$
where Y_i is a $\mathbb{Q}$-Fano variety with stable tangent sheaf with respect to $c_1(Y_i)$ and ω_i is a Kähler-Einstein metric on Y_i.

A few remarks about Theorem 12.4.

• Theorem 12.4 shows that for all "practical aspects" the tangent sheaf of a $\mathbb{Q}$-Fano variety admitting a Kähler-Einstein metric can always be assume to be stable. Moreover, it can be expressed in a purely algebraic way using the notion of K-stability thanks to the solution of the singular Yau-Tian-Donaldson conjecture [**LTW22**] building upon results of [**CDS15a, CDS15b, CDS15c**], [**Li17**], [**BBJ21**] in the smooth case.

• The quasi-étale cover above is needed to split X even when T_X is already split, as we see by taking e.g. $X = (\mathbb{P}^1 \times \mathbb{P}^1)/\langle \iota \times \iota \rangle$ where $\iota : \mathbb{P}^1 \to \mathbb{P}^1$ is the involution $\iota([u:v]) = [u:-v]$.

• It was proved very recently by Braun [**Bra21**, Theorem 2] that the fundamental group of the regular locus of a $\mathbb{Q}$-Fano variety is finite. Relying on that result, one can refine Theorem 12.4 and obtain that the varieties Y_i satisfy the additional property:

$$\pi_1(Y_i^{\mathrm{reg}}) = \{1\}.$$

About the proof of Theorem 12.4.

With Theorem 12.1 and the algebraic integrability of the factors $\mathscr{E}_i$ of T_X in hand, the starting point is the observation that since each foliation $\mathscr{E}_i$ admits a complement inside T_X, $\mathscr{E}_i$ is automatically weakly regular. It turns out that weakly regular foliations have many nice properties. The important fact that needs to be established is that an algebraically integrable, weakly regular foliation on a $\mathbb{Q}$-factorial projective variety with klt singularities is induced by a surjective, equidimensional morphism $X \to Y$, cf [**DGP20**, Theorem 4.6]. When combined with suitable generalisations of other techniques and results in [**Dru21**], this leads to the proof of Theorem 12.4.

13. The Tian-Yau metric as limit of positively curved KE cone metrics

13.1. The Tian-Yau metric. — Let X be a compact Fano manifold of dimension $n \geqslant 2$ endowed with a *smooth* anticanonical divisor $D \subset X$. Note that D is connected by the Lefschetz hyperplane theorem. We denote by L the normal bundle of D, and we fix an hermitian metric h_L on L such that $\theta_D := i\Theta(L, h_L)$ is a Ricci-flat Kähler metric on D (recall that K_D is trivial by adjunction). This yields a function $t := \log |v|_h^2$ on $L \setminus D$ and we also have a connection 1-form η which coincides on each fibre of L with the angular form $d\theta$, and satisfies

$$d\eta = -ip^*\Theta(L, h_L) = -p^*\theta_D,$$

where p is the projection $p : L \to D$. Since $K_X + D \simeq \mathcal{O}_X$ is trivial, we have a global holomorphic n-form Ω on X with a simple pole along D, which can be suitably normalized.

On $\{t < 0\} \subset L \setminus D$, there exists a Ricci-flat metric, complete near D, given by the expression

$$\omega_{TY,L} = \left(\frac{n}{n+1}\right)^{1+\frac{1}{n}} i\partial\bar{\partial}(-t)^{1+\frac{1}{n}}$$

or, after expanding

$$\omega_{TY,L} = (\tfrac{n}{n+1})^{\frac{1}{n}} \big(\tfrac{1}{n}(-t)^{-1+\frac{1}{n}} dt \wedge \eta + (-t)^{\frac{1}{n}} \theta_D\big). \tag{13.1}$$

In [**TY90**], Tian and Yau metric constructed a complete, Kähler metric $\omega_{\mathrm{TY}} = i\partial\bar\partial\varphi_{\mathrm{TY}}$ on $X \setminus D$ satisfying

$$\omega_{\mathrm{TY}}^n = \frac{i^{n^2}}{n+1}\Omega\wedge\overline{\Omega} \tag{13.2}$$

and asymptotic to our Tian-Yau metric $\omega_{TY,L}$ on the normal bundle near D, via a diffeomorphism identifying a neighborhood of D in X with a neighborhood of the zero section in L. Precise asymptotics near D were given by Hein in [**Hei12**]. It follows from (13.2) that $\mathrm{Ric}\,\omega_{\mathrm{TY}} = 0$.

The geometry of $(X \setminus D, g_{\mathrm{TY}})$ is not so standard.

- The distance function $r = d(\cdot, x_0)$ to a fixed point x_0 is comparable to $(-\log|s_D|)^{\frac{n+1}{2n}}$ if s_D is the canonical section of $\mathcal{O}_X(D)$.
- The volume $\mathrm{vol}(B(x_0, r))$ of a ball of radius r behaves like $r^{\frac{2n}{n+1}}$ when $r \to +\infty$, that is less than quadratic, but slightly better than linear, which is the minimal possible growth rate for a complete Ricci flat Kähler metric by a theorem of Yau.
- The volume $\mathrm{vol}(B(x, 1))$ of a ball of radius 1 is of order $r(x)^{-\frac{n-1}{n+1}}$.

13.2. Degeneration of KE cone metrics. — In the same setting as above (X is Fano and $D \in |-K_X|$ is a smooth divisor), we see that the $\mathbb{Q}$-line bundle $-(K_X + (1-\beta)D) = \beta D$ is ample for any $\beta > 0$. A result of Berman [**Ber13**] (later generalized by Song-Wang [**SW16**]) asserts that there exists $\beta_0 > 0$ such that for any $0 < \beta < \beta_0$, there exists a unique Kähler metric ω_β on $X \setminus D$ such that $\mathrm{Ric}\,\omega_\beta = \omega_\beta$ and ω_β has cone singularities with cone angle $2\pi\beta$ along D, i.e. ω_β solves

$$\mathrm{Ric}\,\omega_\beta = \omega_\beta + (1-\beta)[D] \tag{13.3}$$

in the sense of currents.

The existence of such a metric had been conjectured by Donaldson [**Don12**, § 6] in relation with his program to prove that a K-stable Fano manifold admits a Kähler-Einstein metric by using the continuity path $\mathrm{Ric}\,\omega_t = t\omega_t + (1-t)[D]$ involving metrics with cone singularities. He also predicted that the (conjectural then) ω_β would actually converge to the Ricci flat complete Kähler metric ω_{TY} constructed by Tian and Yau in [**TY90**].

If $n = 1$, then the metrics ω_β on $\mathbb{P}^1 \setminus \{0, \infty\}$ are completely explicit, given by the expression $\omega_\beta = \frac{\beta^2 i dz\wedge d\bar z}{|z|^{2(1-\beta)}(1+|z|^{2\beta})^2}$ and one sees immediately that $\beta^{-2}\omega_\beta$ converges locally smoothly to the cylinder $\omega_{\mathrm{cyl}} = \frac{idz\wedge d\bar z}{4|z|^2}$ while $(\mathbb{P}^1, \omega_\beta)$ converges in the Gromov-Hausdorff sense to the interval $([0, \frac{\pi}{2}], dt^2)$ (set $r = |z|^\beta$ to that $g_\beta = \frac{dr^2 + \beta^2 r^2 d\theta^2}{(1+r^2)^2}$ and reparametrize by $t = \tan^{-1}(r)$), cf also [**RZ20**].

With Olivier Biquard, we showed that this phenomenon happens in any dimension:

Theorem 13.1 **([BG22]).** — *Let X be a Fano manifold of dimension n and let $D \in |-K_X|$ be a smooth anticanonical divisor. Then up to a rescaling factor, the conic KE metrics ω_β solving* (13.3) *for small β converge to the Tian-Yau metric:*

$$\beta^{-1-\frac{1}{n}}\omega_\beta \underset{\beta\to 0}{\longrightarrow} \omega_{\mathrm{TY}}$$

in $C^\infty_{\mathrm{loc}}(X \setminus D)$. Moreover, we have precise asymptotics of ω_β near D when $\beta \to 0$.

Finally fix a point $p \in D$, denote g_β the Riemannian metric associated to the Kähler form ω_β and consider the renormalized volume forms $\nu_\beta = \frac{\mathrm{dvol}_{g_\beta}}{\mathrm{vol}_{g_\beta}(X)}$. Then the spaces $(X, g_\beta, p, \nu_\beta)$ converge in the measured Gromov-Hausdorff sense to the interval

$$\big([0, \tfrac{\pi}{2}], g_\infty = \tfrac{2}{n+1} ds^2, 0, \nu_\infty = d(\cos^{\frac{2n}{n+1}} s)\big).$$

A few remarks:

- The fibers of the collapsing to an interval are the normal circle bundle of the divisor D. The two endpoints of the interval correspond respectively to the conical divisor D itself and to the Tian-Yau metric. Over interior points of the interval, the fibres have two speeds of collapsing: speed β for the circle directions and $\sqrt{\beta}$ for the divisor directions.
- Several recent papers study cases of collapsing of Ricci flat Kähler metrics to an interval, for K3 surfaces [**HSVZ22**] or in higher dimension [**SZ19**]. Our theorem probably gives the first general example of collapsing of Kähler-Einstein metrics with positive Ricci: of course this is made possible by the presence of a cone angle going to zero.
- In the process of the proof, we construct the Kähler-Einstein metrics ω_β for small β, therefore recovering Berman's result.

Strategy of proof of Theorem 13.1.

Step 1. *Construction of a Calabi cone metric.*

On the normal bundle L, we look for a potential $\varphi_\beta = \varphi_\beta(t)$ such that the induced Kähler metric $\omega_{\beta,L} := i\partial\bar\partial\varphi_\beta$ is a Kähler-Einstein cone metric on $L \setminus D$ with cone angle $2\pi\beta$ along D, and Ricci constant 1, i.e.

$$\mathrm{Ric}\,\omega_{\beta,L} = \omega_{\beta,L} + (1-\beta)[D].$$

This equation can be reduced to an ODE which is solvable explicitly in terms of some integral, and we get a solution $\varphi_\beta(t)$ defined for all $t < 0$. Moreover, we have the crucial relation

$$\varphi_\beta(t) = \varphi_1(\beta t)$$

which allows us to understand completely the asymptotic behavior of $\omega_{\beta,L}$ when $\beta \to 0$. This also indicates that the variable $u := \beta t \in (-\infty, 0)$ will be better suited for what will follow. Using that variable, the Riemannian metric $g_{\beta,L}$ associated to $\omega_{\beta,L}$ becomes

$$g_{\beta,L} = 2\varphi_1''(u)(\tfrac14 du^2 + \beta^2\eta^2) - \beta\varphi_1'(u) g_D \tag{13.4}$$

and we have the asymptotic when $u \to -\infty$:

$$\varphi_1'(u) = -\sigma + \frac{1}{n+1}e^u + O(e^{2u})$$

from which follows, taking $r = e^{\frac{u}{2}} \in (0,1)$,

$$\begin{aligned} g_{\beta,L} &= \tfrac{2}{n+1} e^u \big(\tfrac14 du^2 + \beta^2\eta^2\big) + \beta g_D + O(e^u) \\ &= \tfrac{2}{n+1}\big(dr^2 + \beta^2 r^2\eta^2\big) + \beta g_D + O(r^2), \end{aligned}$$

where the $O(r^2)$ is with respect to $g_{\beta,L}$ and is uniform with respect to β. From this, one understanding clearly the collapsing behavior of $g_{\beta,L}$ when $\beta \to 0$: the divisor is collapsed at speed $\sqrt{\beta}$ while the normal circle directions are collapsing at speed β.

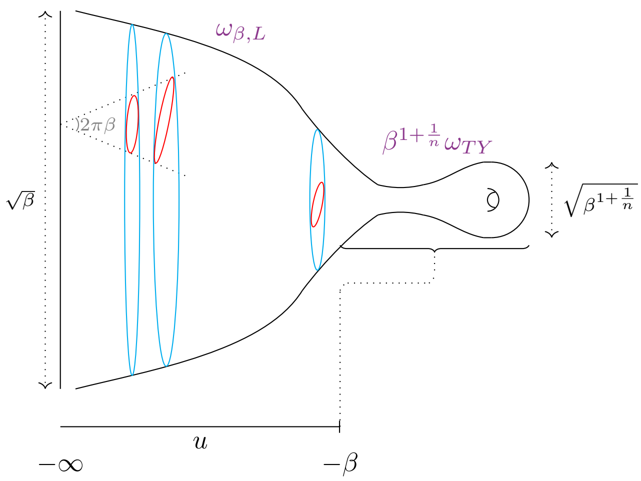
Step 2. *The gluing.*

We will use a canonical diffeomorphism $\Phi : U \to U_L$ identifying a neighborhood $U \subset X$ of D to a neighborhood $U_L \subset L$ of the zero section and which is holomorphic along the disks normal to D. In particular, it satisfies $i\partial\bar\partial\Phi^*\varphi_{\beta,L} = \Phi^*(i\partial\bar\partial\varphi_{\beta,L})$. We can then use a cut-off function (depending on β) to define a new potential φ_β on $X \setminus D$ such that

$$\varphi_\beta = \begin{cases} \Phi^*\varphi_{\beta,L} & \text{if } \ u < 2u_\beta/ \\ \beta^{1+\frac{1}{n}}\varphi_{\mathrm{TY}} & \text{if } \ u > u_\beta/2 \end{cases}$$

where $u = \beta t$ on U (rather the pull back of t by Φ, which we truncate on $\{-2 < t < -1\}$ to be identically -1 near ∂U and then extend by -1 on $X \setminus U$). Moreover, we choose $u_\beta = -\beta^\mu$ for some fixed $\mu \in (0,1)$ which we will at the very end of the argument take close to 1 – that is, we glue the Calabi metric to the Tian-Yau metric "rather far away from D in the Tian-Yau part".

We denote by $\omega_\beta := i\partial\bar\partial\varphi_\beta$ the metric obtained this way, which patches a small modification of the Calabi metric to the Tian-Yau metric. The asymptotic behavior of (X, ω_β) is summarized by the picture below.



Step 3. *Analysis of the linearization.*

The Kähler metric ω_β is certainly not Einstein (it is Ricci flat away from D, but is closed to having Ric $= 1$ near D); the strategy is deform ω_β into a (actually, the) Kähler-Einstein cone metric $\widehat{\omega}_\beta := \omega_\beta + i\partial\bar\partial\widehat{\varphi}_\beta$ for some function $\widehat{\varphi}_\beta$ under control, e.g. all of whose derivatives are a $o(\beta^{1+\frac{1}{n}})$ on a given compact subset of $X \setminus D$.

We look at the operator

$$P_\beta(\varphi) := \log \frac{(\omega_\beta + i\partial\bar\partial\varphi)^n}{i^{n^2}\Omega \wedge \overline{\Omega}} + (\varphi_\beta + \varphi) - c_\beta$$

where the constant c_β is the constant obtained for the model Calabi metric $g_{\beta,L}$, that is $c_\beta = (n+1)\log\beta - \log(n+1)$. We can estimate $P_\beta(0)$ explicitly, and then hope that it is small enough and $(dP_\beta)_0^{-1}$ not too big so that one can find a solution $P_\beta(\varphi)+0$ in a ball around 0 in some suitable functional space.

The linerarization of P_β at 0 is simply $\Delta_{\omega_\beta}+1 : \mathcal{C}^{2,\alpha}\to\mathcal{C}^\alpha$, and the remaining on the proof (also the bulk of it) relies on estimating its norm in some suitably weighted Hölder spaces, that is estimating the optimal constant C_β such that for all $f\in\mathcal{C}^{2,\alpha}$, we have

$$\|f\|_{\mathcal{C}^{2,\alpha}} \leqslant C_\beta \|(\Delta_{\omega_\beta}+1)f\|_{\mathcal{C}^\alpha}. \tag{13.5}$$

Since the action of $\Delta_{\omega_\beta}+1$ on constants is just the identity, it comes down to showing (13.5) with $C_\beta\simeq 1$ for functions f satisfying $\int_X f d\mathrm{vol}_{g_\beta}=0$

Assume for the time being that we can prove a Schauder estimate independent of β

$$\|f\|_{\mathcal{C}^{2,\alpha}} \leqslant C\left(\|f\|_{C^0}+\|\Delta_{\omega_\beta} f\|_{\mathcal{C}^\alpha}\right). \tag{13.6}$$

Then, contradicting (13.5) would provide a sequence $f_k=f_{\beta_k}$ with $\beta_k\to 0$ such that

- $\int_X f_k d\mathrm{vol}_{g_{\beta_k}}=0.$
- $\|f_k\|_{C^{2,\alpha}}=1$, $\|(\Delta_{\omega_{\beta_k}}+1)f_k\|_{\mathcal{C}^\alpha}\to 0.$
- $\|f_k\|_{\mathcal{C}^0}$ remains bounded away from zero.

The general idea is to pick x_k where $f_k(x_k)=\eta>0$ and run a blow-up argument to extract a non-zero limit f of f_k on some Gromov-Hausdorff limit of (X, g_{β_k}, x_k) which will then be endowed with a Bakry-Emery Laplacian Δ and satisfy $(\Delta+1)f=0$ which will be prevented by the geometry of X and the integral normalization condition. There are actually three cases to consider, depending on whether x_k is asymptotically in the zone where ω_β is either $\omega_{\beta,L},\omega_{\mathrm{TY}}$ or a gluing of both. For instance in the case where one has $u(x_k)<\eta<0$, one can see that f_k converges to a function f defined on some neighborhood of the zero section in L. It turns out that f depends on u only because the norm $\|df_\beta\|_{C^0}$ involves a factor β^{-1} in the circle direction or $\beta^{-1/2}$ in the divisor D direction. This implies that $f=f(u)$ satisfies

$$\frac{f''(u)}{\varphi_1''(u)}+(n-1)\frac{f'(u)}{\varphi_1'(u)}+f(u)=0. \tag{13.7}$$

Moreover, one can check that the normalizing condition translates into

$$\int_{-\infty}^0 f(u)\varphi_1''(u)\varphi_1'(u)^{n-1}du=0. \tag{13.8}$$

The function $\varphi_1'(u)$ is an obvious solution of (13.7), it corresponds to the dilation vector field in the bundle L. It satisfies $\varphi_1'(u)\to -1$ when $u\to-\infty$, while the other solution can be shown to be equivalent to u near $-\infty$, which is ruled out by the $\mathcal{C}^0$ bound on f_k. Therefore we see that up to a constant we must have $f(u)=\varphi_1'(u)$, which gives a contradiction with (13.8).

Step 4. *Schauder estimate.*
It remains to establish (13.6), which will be achieved in several steps, each of them quite technical and lengthy. We will just give a brief overview of the strategy.

The first step is to establish a Schauder estimates of the form

$$\|f\|_{\mathcal{C}^{2,\alpha}} \leqslant C\left(\|f\|_{C^0}+\|\Delta_{\omega_\beta} f\|_{\mathcal{C}^\alpha}\right)$$

with respect to the metric

$$(dr^2 + r^2\beta^2 d\theta^2) + g_{\mathbb{C}^{n-1}}$$

and valid on a ball of radius one centered on the divisor. For fixed β, the estimate was well-known (see e.g. [**GS21b, GY21, DE21**]), but the novelty is to get uniformity with respect to the angle degenerating to zero, i.e. when the circle directions collapse. Note that since we work in a small regime, we can actually control the full $\mathcal{C}^{2,\alpha}$ norm of f and not just the mixed derivatives.

With this first step in hand, we can relatively easily deduce Schauder estimates for $\frac{1}{\beta}g_\beta$; this is a scale at which the circle collapses but the divisor does not collapse. Indeed, near the divisor $\frac{1}{\beta}g_\beta$ is essentially the one considered in the first step. Moreover, $\frac{1}{\beta}g_\beta$ turns out to have bounded curvature so that we can obtain Schauder estimates from standard arguments far from the divisor D, and conclude this second step.

The next and final step is to work at the scale of $g_{\beta,L}$, where both the circle directions and the divisor collapse (at different speed though). In our problem we need estimates at this scale, on balls of fixed radius, say ϱ, for $g_{\beta,L}$; this corresponds to balls of larger and larger radius $\frac{\varrho}{\sqrt{\beta}}$ in the geometry of $\frac{1}{\beta}g_{\beta,L}$; we obtain these estimates from a global estimate on some limit of $\frac{1}{\beta}g_{\beta,L}$, relying notably on Fourier decomposition and separating the modes.

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