Exercises for the course "Topics in Complex Algebraic Geometry"

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1 Complex manifolds, differential forms

Exercise 1.1. Let X be a compact complex curve. Let $f : X \to Y$ be a holomorphic map where Y has dimension at least two. Show that f is not surjective. (hint 1: use Sard's theorem. hint 2 : alternatively, use the normal form of holomorphic functions of one variable to show that the image of X is locally a finite ramified cover of the disk in $\mathbb{C}$).

Exercise 1.2. Let $U \subset \mathbb{C}^n$ be an open subset satisfying $H^1_{\mathrm{dR}}(U, \mathbb{C}) = 0$. Let $\varphi \in C^\infty(U)$ such that $\partial\bar{\partial}\varphi = 0$. Show that there exists $f \in \mathcal{O}(U)$ such that $\varphi = \mathrm{Re}(f)$.

2 Line bundles, Picard and Albanese varieties

Here and unless otherwise stated, X is a compact Kähler manifold.

Exercise 2.1. Show that if both $H^0(X, L) \neq 0$ and $H^0(X, L^{-1})$ are not trivial, then L is trivial.

Exercise 2.2. Let L be a torsion line bundle. Show that L is trivial if, and only if $H^0(X, L) \neq 0$.

Exercise 2.3. Take $L \in \mathrm{Pic}^\circ(X)$. Show that given $m \in \mathbb{Z}$, there exists $N \in \mathrm{Pic}^\circ(X)$ such that $L \simeq N^{\otimes m}$.

Exercise 2.4. Give a "cocycle" proof of the Lefschetz theorem on $(1,1)$-classes.

Exercise 2.5. Let X be a Riemann surface genus at least two. Using Exercise 1.1, show that the Albanese map of X is not surjective. Deduce that for $p \in X$ and $L \in \mathrm{Pic}^\circ(X)$ general, the line bundle $\mathcal{O}_X(p) \otimes L$ is ample but has no sections.

Exercise 2.6. Compute the Picard group of an elliptic curve. Given an example of such a curve admitting a torsion, non-trivial line bundle.

Exercise 2.7. Set $q = \dim H^1(X, \mathcal{O}_X)$. Show that $\mathrm{Pic}^\circ(X)_{\mathrm{tor}} \simeq (\mathbb{Q}/\mathbb{Z})^{2q}$.

Exercise 2.8. Let L be a line bundle such that $c_1(L) = 0$.

1. Show that there exists an hermitian metric h on L such that $\nabla_h^{1,0}$ is a holomorphic connection. Here, ∇_h is the Chern connection of L and being a holomorphic connection means that for any local section $s \in H^0(U, L)$, one has $\nabla_h^{1,0} s \in H^0(U, L \otimes \Omega_X^1)$.

2. Show that L admits local trivializing sections s such that $\nabla_h^{1,0} s = 0$.

3. Show that one can choose transition functions of L that are locally constant. Deduce that the pullback of L to the universal cover of X is trivial.

Exercise 2.9. Let $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ be a representation and let $\pi_1(X)$ acts on $\tilde{X} \times \mathbb{C}^r$ diagonally (via the natural action of $\pi_1(X)$ on $\tilde{X}$ and ρ on $\mathbb{C}^r$. Set $E := (\tilde{X} \times \mathbb{C}^r)/\pi_1(X)$.

1. Show that E admits a structure of holormorphic vector bundle over X.

2. Show that one can choose transition functions for E that are locally constant.

3. In case $r = 1$, show that $c_1(E) = 0$.

Exercise 2.10. Let L be a line bundle such that $c_1(L) = 0$. Denote by $\pi : \tilde{X} \to X$ the universal cover of X.

1. Show that the map $\pi^* : H^1(X, \mathcal{O}_X) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is the zero map (use the isomorphism $H^1(X, \mathcal{O}_X) \simeq H^{0,1}(X)$ and represent classes by harmonic ones).

2. Deduce that $\pi^* L$ is trivial.

Exercise 2.11. Let L be a line bundle on a projective manifold of dimension n. Show that $h^0(X, kL) = O(k^n)$ as $k \to +\infty$. (hint: one can argue by induction and show that $h^0(X, kL) \leqslant k h^0((kL)|_H) + h^0(k(L - H))$).

Exercise 2.12. Let X be a compact Kähler manifold such that $H^{1,0}(X) = 0$. Show (without using the Albanese map) that there is no non-constant map $X \to T$ to a complex torus. (hint: the map $\pi_1(X) \to \pi_1(T)$ factors through $H_1(X, \mathbb{Z})/\mathrm{tor}$).

3 Intersection numbers

Exercise 3.1. Let L be a line bundle such that $c_1(L) = 0$. Show that $H^0(X, L) = 0$ unless L is trivial.

Exercise 3.2. Let $f : X \to C$ be a surjective map with connected fibers from a compact Kähler surface to a compact Riemann surface. Let C° be the complement of the critical locus of f. Given $c_i \in C^\circ$ ($i = 1, 2$) set $D_i = f^{-1}(c_i)$.

1. Assuming $C = \mathbb{P}^1$, show that $\mathcal{O}_X(D_1)$ is isomorphic to $\mathcal{O}_X(D_2)$.

2. In general, show that $c_1(D_1) = c_1(D_2)$.

3. Show that $c_1(D_1)^2 = 0$.

4 Lefschetz theorem

Exercise 4.1. Show that the only complex tori that can be embedded as a complete intersection in the projective space are elliptic curves.

Exercise 4.2. Show that a hypersurface in $\mathbb{P}^n$ with $n \geqslant 3$ does not admit any non-zero one forms. What if $n = 2$?

Exercise 4.3. Give an example of an ample hypersurface $Y \subset X$ such that $H^{n-1}(X, \mathbb{C}) \to H^{n-1}(Y, \mathbb{C})$ is not an isomorphism, where $n = \dim X$. E.g. one can look at H^2 of a quartic surface.

5 Canonical bundle

Exercise 5.1. Let f be a holomorphic function on $\mathbb{C}^n$ such that df does not vanish along $X = (f = 0)$. Show that the forms

$$(-1)^i \frac{1}{\partial_i f} dz_1 \wedge \dots \widehat{dz_i} \cdots \wedge dz_n|_{X \cap (\partial_i f \neq 0)}$$

glue. In particular K_X is trivial.

Generalize this construction to the case of hypersurfaces in $X \subset \mathbb{P}^n$ of degree $n + 1$ (and then arbitrary degree, by constructing meromorphic section of K_X.

Exercise 5.2. Take the Fermat hypersurface $Y = \{\sum z_i^p = 0\} \subset \mathbb{P}^{n+1}$ with $p = n + 2$. Show that it is smooth, has trivial canonical bundle, and has a free action of $G = \mathbb{Z}/p\mathbb{Z}$. Is the canonical bundle of $X = Y/G$ trivial?

Exercise 5.3. Let $X_d \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d.

1. Compute $(K_{X_d}^n)$.
2. Compute $\dim H^0(X_d, K_{X_d})$.

6 Blow-ups

Exercise 6.1. Let X be a complex projective manifold and let $\pi : \tilde{X} \to X$ be a blow up of a smooth submanifold with exceptional divisor E.

1. Show that π induces an isomorphism $\pi^* : H^1(X, \mathbb{C}) \to H^1(\tilde{X}, \mathbb{C})$.
2. Show that $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ is injective and that $\mathrm{Pic}(\tilde{X}) \simeq \pi^*\mathrm{Pic}(X) \oplus \mathbb{Z}\mathcal{O}_X(E)$.

Exercise 6.2. Compute (E^2) where E is the exceptional divisor of the blow up of a point on a surface.

Exercise 6.3. Let X be a complex manifold, let $p \in X$ and let $X_1 := \mathrm{Bl}_p X$ with induced map $\pi_1 : X_1 \to X$. Let q be a point on the exceptional divisor E_1 of π_1 and let $X_2 = \mathrm{Bl}_q X_1$. Denote by $\pi_2 : X_2 \to X_1$ the blow-up map and let E_2 be the exceptional divisor of π_2.

1. Compute $\pi_2^* E_1$ in terms of E_2 and E_1', the strict transform of E_1.

2. Compute the canonical bundle of X_2 in terms of the one of X.

3. Compute the intersection matrix $(E_i \cdot E_j)_{1 \leqslant i,j \leqslant 2}$ and show that it is negative definite. One can use Exercise 6.2.

Exercise 6.4. Let $C = (y^2 - x^3 = 0) \subset \mathbb{C}^2$ and let $\pi : X \to \mathbb{C}^2$ be the blow up of the origin. Compute the strict transform of C, its intersection points with the exceptional divisor and the tangencies at these points.

Exercise 6.5. Let $X = (x^2 + y^2 + z^{n+1} = 0) \subset \mathbb{C}^3$ for some integer $n \geqslant 1$. Consider the blow up of the origin in $\mathbb{C}^3$ with exceptional divisor E.

1. Compute the strict transform X' of X and determine whether it is smooth or not.

2. Compute $X' \cap E$.

Exercise 6.6. Let $X = (xy = zt) \subset \mathbb{C}^4$. for some integer $n \geqslant 1$. Consider the blow up of the origin in $\mathbb{C}^3$ with exceptional divisor E.

1. Show that $X \simeq (\sum_{i=1}^4 z_i^2 = 0) \subset \mathbb{C}^4$.

2. Consider the blow-ups of $\mathbb{C}^3$ along 0 and $(y = z = 0)$ respectively. Compute the strict transform X_1, X_2 of X under each blow-up.

3. Show that the exceptional loci of $X_i \to X$ have different dimension for $i = 1, 2$.

Exercise 6.7. Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d and let $\pi : X \to \mathbb{C}^{n+1}$ be the blow up of the origin with exceptional divisor E. Write $C = (f = 0) \subset \mathbb{C}^{n+1}$ and denote by C' the strict transform of C by π. It induces a birational map $p : C' \to C$. Show that $F := C' \cap E$ is isomorphic to the projective hypersurface $(f = 0) \subset \mathbb{P}^n$. Compute the normal bundle of $F \subset C'$ and its top intersection number.

7 Cyclic coverings

Exercise 7.1. Let $D \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. Given $q|d$, show that there exists a cover $Y \to \mathbb{P}^n$ of degree q totally ramified along D. Find a necessary and sufficient condition on n, d, q so that K_Y is trivial.

Exercise 7.2. Let $D \subset \mathbb{P}^2$ be the union of two lines meeting. Explain why there exists a cover $Y \to \mathbb{P}^2$ of degree 2 ramified above D. Is Y smooth?

Exercise 7.3. Let X be a compact Kähler manifold, let $D \subset X$ be a smooth divisor and let $d \geqslant 1$ be an integer. Show that there exists $L \in \mathrm{Pic}(X)$ such that $\mathcal{O}_X(D) \simeq L^{\otimes d}$ if and only if $c_1^{\mathbb{Z}}(\mathcal{O}_X(D)) \in H^2(X, \mathbb{Z})$ is divisible by d. If that happens, show that there exists a cyclic covering of degree d totally branched over D.

8 Miscellaneous

Exercise 8.1. Show that if L is a line bundle on a compact complex manifold such that $\mathbb{B}(L)$ is empty, then the image $\Phi_L(X)$ of $\Phi_L : X \to \mathbb{P}(H^0(X, L)^*)$ is not contained in an hyperplane.

Exercise 8.2. Using Riemann's bilinear relations, show that any one-dimensional complex torus is projective.

Exercise 8.3. Let S a compact Kähler surface such that there exists a surjective holomorphic map $f : S \to C$ where C is a compact Riemann surface and for $c \in C$ general, $S_c := f^{-1}(c)$ is isomorphic to $\mathbb{P}^1$.

1. Show that $K_S|_{S_c} \sim K_{S_c}$.

2. Show that S is projective.