ERRATUM FOR THE ARTICLE "TORIC PLURISUBHARMONIC FUNCTIONS AND ANALYTIC ADJOINT IDEAL SHEAVES"

by

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Abstract. — In this erratum, we explain a mistake in [**Gue12**], pointed out to us by Mario Chan, where it was wrongly claimed that on a complex manifold X, the algebraic and ideal adjoint ideal sheaves associated to an SNC divisor D and a coherent ideal sheaf $\mathfrak{a}$ coincide. Although this is false is general, we show that the result holds if the divisor D is smooth.

The definition stated in [**Gue12**, Definition 2.1] of the algebraic adjoint ideal attached to an ideal $\mathfrak{a}$ and a reduced divisor D on a complex manifold X is unfortunately incorrect, since the sheaf in that definition may depend on the chosen log resolution of the ideal generated by $\mathfrak{a}$ and $\mathcal{O}_X(-D)$. The correct definition, as stated e.g. in [**Tak10**] or [**Eis10**] in a higher degree of generality, is as follows

Definition 0.1. — Let $\mathfrak{a} \subset \mathcal{O}_X$ be a non-zero coherent ideal sheaf on a smooth complex variety X, $c > 0$ a real number, and D a reduced divisor on X such that $\mathfrak{a}$ is not contained in any ideal $\mathscr{I}_{D_i}$ of D_i an irreducible component of D. Let $\mu : \widetilde{X} \to X$ be a log resolution such that $\widetilde{X}$ is smooth, the strict transform $D' = \mu_*^{-1} D$ of D is smooth, and $\mu^{-1}\mathfrak{a} = \mathcal{O}_{\widetilde{X}}(-F)$ where F is a divisor on $\widetilde{X}$ such that $F + D' + \mathrm{Exc}(\mu)$ has simple normal crossings support. Then, the ideal sheaf

$$\mathrm{Adj}(\mathfrak{a}^c, D) := \mu_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X} - \lfloor c \cdot F \rfloor - \mu^* D + D')$$

is independent of the resolution and called the adjoint ideal sheaf associated ot $(\mathfrak{a}, D)$.

In [**Gue12**, Definition 2.1], we had not required the condition that D' be smooth, in which case the sheaf one obtains depends on the resolution as one sees by taking e.g. $X = \mathbb{C}^2$, $\mathfrak{a} = \mathcal{O}_X$ and $D = (z_1 z_2 = 0)$. Then $\mu_1 = \mathrm{Id}_{\mathbb{C}^2}$ and $\mu_2 = \mathrm{Bl}_0(\mathbb{C}^2) \to \mathbb{C}^2$ yield two different ideals (respectively $\mathcal{O}_X$ and $\mathfrak{m}_0$).

As pointed out by Mario Chan (see [**Cha21**, Example 6.3.1]), this example shows that the algebraic adjoint ideal associated to $(\mathfrak{a}, D)$ where D is SNC does not coincide with the analytic adjoint ideal defined in [**Gue12**, Definition 2.10], so that [**Gue12**, Proposition 2.11] is incorrect. The error lies in the proof of [**Gue12**, Lemma 2.12] where at the beginning of the proof we assert that $a_k > 0$ while this is not true in general. However, the equality of the two sheaves holds when D is smooth, as we explain below (see also [**Cha21**, Theorem 5.2.5] for an alternative approach).

Proposition 0.2. — *Let D be a smooth hypersurface on a complex manifold X, $\mathfrak{a}$ a coherent analytic ideal sheaf on X not containing any of the ideals of the components of D, $c > 0$ a real number and $\varphi_{c\cdot\mathfrak{a}}$ be a psh function attached to $\mathfrak{a}^c$. Then the following equality of sheaves holds:*

$$\mathcal{A}dj_D(\varphi_{c\cdot\mathfrak{a}}) = \mathrm{Adj}(\mathfrak{a}^c, D).$$

Proof. — The argument borrows most of the computations in the proof of [**Gue12**, Proposition 2.11].

The problem is local, so one can assume that $X \subset \mathbb{C}^n$ is an open subset, that D is irreducible given by $f_D = 0$ for some holomorphic function f_D on X. Let $\mu : X' \to X$ be a modification as in Definition 0.1; we can assume that μ is isomorphic outside the support $V(\mathfrak{a}) := \mathrm{Supp}(\mathcal{O}_X/\mathfrak{a})$. We write

$$K'_X + D' = \mu^*(K_X + D) + \sum_{j \in J} a_j E_j$$

where $(E_j)_{j\in J}$ are the exceptional divisors of μ and $a_j \in \mathbb{Z}$. Actually, it follows from the smoothness of D that the a_j's satisfy $a_j \geqslant 0$ thanks to [**Kol97**, Lemma 3.11], but we won't be using it.

Write $\mu^*\mathfrak{a} = \mathcal{O}_{X'}(-F)$ where $F = \sum_{i\in I} c_i E_i$ where $I \supset J$, $c_i \in \mathbb{N}$ and $D' + \sum_{i \in I} E_i$ is SNC. The E_i's with $i \in I \setminus J$ correspond to the divisorial components of $V(\mathfrak{a})$. The assumption on $\mathfrak{a}$ imposes that

$$\forall i \in I, \quad D' \neq E_i. \tag{1}$$

Moreover, since μ is isomorphic outside $V(\mathfrak{a})$, every function in $\mu^*\mathfrak{a}$ vanishes along every exceptional divisor. In other words,

$$\forall i \in I, \quad c_i > 0. \tag{2}$$

If f is a germ of holomorphic function defined on a sufficiently small neighborhood U of 0, we have to compute the following expression:

$$\int_U \frac{|f|^2 e^{-2(1+\epsilon)\varphi}}{|f_D|^2 \log^2|f_D|} dV = \int_{\mu^{-1}(U)} \frac{|f\circ\mu|^2 e^{-2(1+\epsilon)\varphi\circ\mu}}{|f_D\circ\mu|^2 \log^2|f_D\circ\mu|} |J_\mu|^2 \, dV'. \tag{3}$$

We can cover $\mu^{-1}(U)$ with finitely many coordinate charts $(U', z_1, \dots, z_n)$. The charts that do not intersect D' can be treated similarly to the ones intersecting D', so we will focus on the latter case. We can then choose the coordinates such that $D' \cap U = (z_1 = 0)$. Moreover, we can ensure thanks to (1) that there is an injection $\sigma : \{2, \dots, m\} \to I$ such that the only components of E intersecting U' are the $(E_{\sigma(i)})_{2\leqslant i \leqslant m}$ and that these components are given by $E_{\sigma(i)} \cap U' = (z_i = 0)$. For convenience, one will identify a_i and $a_{\sigma(i)}$, and set

$$a_1 := -1, \quad c_1 = 0.$$

Thanks to Parseval's theorem, if a function f is such that the right hand side of (3) is finite, then all monomials in the Taylor expansion of f satisfy the same property. So there is no loss of generality in supposing that $f \circ \mu = \prod_{i=1}^n z_i^{d_i}$. Thus, up to a non-zero multiplicative constant, (3) can be expressed as a sum of integrals of the form :

$$\int_{U'} \frac{\prod_{i=1}^m |z_i|^{2(d_i - (1+\epsilon)cc_i + a_i)}}{\log^2(|z_1| \prod_{i=2}^m |z_i|^{b_i})} dV'$$

where U' is contained in a small polydisk in $\mathbb{C}^n$ and the integers $b_i \geqslant 0$ are defined by the identity $\mu^* D = D' + \sum_{i=2}^m b_i E_{\sigma(i)}$ holding on U'.

Set $\lambda_i(\epsilon) = 2(d_i - (1+\epsilon)cc_i + a_i) + 1$ for all $1 \leqslant i \leqslant m$, and changing to polar coordinates leads us to estimating the following integral, on a neighborhood V of 0 in $\mathbb{R}_+^m$:

$$I(\epsilon) = \int_V \frac{\prod_{i=1}^m x_i^{\lambda_i(\epsilon)}}{\log^2(x_1 \prod_{b_i > 0} x_i)} dx_1 \cdots dx_m$$

and V some small neighborhood of 0 in $\mathbb{R}_+^m$.

Now, $f \in H^0(X, \mu_* \mathcal{O}_{X'}(K_{X'/X} - [c \cdot F] - \mu^* D + D'))$ if and only if for all U' as above, we have $d_i + a_i - \lfloor c c_i \rfloor \geqslant 0$. But for any real number $x \geqslant 0$, we have $\lfloor (1+\epsilon)x \rfloor = \lfloor x \rfloor$ for $\epsilon > 0$ small enough (any $\epsilon < (\lfloor x \rfloor + 1)/x - 1$ does the job). Therefore,

$$f \in H^0(X, \mathrm{Adj}(\mathfrak{a}^c, D)) \Longleftrightarrow \exists\, \epsilon > 0, \forall\, i \in \{1, \ldots, m\},\ \lambda_i(\epsilon) \geqslant -1. \tag{4}$$

Note that

$$\lambda_1(\epsilon) = 2d_1 - 1 \geqslant -1, \tag{5}$$

so that we only have to consider indices i satisfying $i \geqslant 2$. The conclusion now follows from the lemma below, give or take the fact that we have only worked on charts U' such that $D' \cap U' \neq \emptyset$. The argument to deal with the other charts is entirely similar, yet a bit easier so we don't reproduce it here. ☐

Lemma 0.3. — *Fix $\epsilon > 0$. Then the integral $I(\epsilon')$ converges for all $0 < \epsilon' < \epsilon$ if and only if for any $i \in \{2, \ldots, m\}$, we have $\lambda_i(\epsilon) \geqslant -1$.*

Proof. — If $I(\epsilon')$ converges, then the usual criterion which determines the integrability near 0 of $x^\alpha \log^\beta x$ shows that $\lambda(\epsilon') \geqslant -1$. We can then let ϵ' go to ϵ to obtain one assertion in the claim. Conversely, we suppose that for all $i \geqslant 2$, we have $\lambda_i(\epsilon) \geqslant -1$. Thanks to (5) and the following identity holding for any $x > 0$

$$\int_{]0,\delta]} \frac{y^{-1}}{\log^2(xy)} dy = \frac{1}{-\log(\delta x)}$$

we get that up to a multiplicative constant,

$$I(\epsilon) \leqslant \int_W \frac{\prod_{i=2}^m x_i^{\lambda_i(\epsilon)}}{-\log(\prod_{b_i>0} x_i)} dx_2 \cdots dx_m$$

for W some small neighborhood of 0 in $\mathbb{R}_+^{m-1}$.

To take care of that integral, we observe that for $i \geqslant 2$, we have $c_i > 0$ as a consequence of (1)-(2). This implies that $\lambda_i(\epsilon') > -1$ for any $\epsilon' < \epsilon$. The convergence of the integral is now immediate. ☐

Remark 0.4. — The above proof generalizes to the case where D is merely a normal hypersurface of X.

Acknowledgement. The author would like to thank Mario Chan for pointing out the inaccuracy in [Gue12].

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