Exercises for the course "Topics in Complex Algebraic Geometry"

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1 Complex manifolds, differential forms

Exercise 1.1. Let X be a compact complex curve. Let $f : X \to Y$ be a holomorphic map where Y has dimension at least two. Show that f is not surjective. (hint 1: use Sard's theorem. hint 2 : alternatively, use the normal form of holomorphic functions of one variable to show that the image of X is locally a finite ramified cover of the disk in $\mathbb{C}$).

Exercise 1.2. Let $U \subset \mathbb{C}^n$ be an open subset satisfying $H^1_{\mathrm{dR}}(U, \mathbb{C}) = 0$. Let $\varphi \in C^\infty(U)$ such that $\partial\bar{\partial}\varphi = 0$. Show that there exists $f \in \mathcal{O}(U)$ such that $\varphi = \mathrm{Re}(f)$.

2 Line bundles, Picard and Albanese varieties

Here and unless otherwise stated, X is a compact Kähler manifold.

Exercise 2.1. Show that if both $H^0(X, L) \neq 0$ and $H^0(X, L^{-1})$ are not trivial, then L is trivial.

Exercise 2.2. Let L be a torsion line bundle. Show that L is trivial if, and only if $H^0(X, L) \neq 0$.

Exercise 2.3. Take $L \in \mathrm{Pic}^\circ(X)$. Show that given $m \in \mathbb{Z}$, there exists $N \in \mathrm{Pic}^\circ(X)$ such that $L \simeq N^{\otimes m}$.

Exercise 2.4. Give a "cocycle" proof of the Lefschetz theorem on $(1, 1)$-classes.

Exercise 2.5. Let X be a Riemann surface genus at least two. Using Exercise 1.1, show that the Albanese map of X is not surjective. Deduce that for $p \in X$ and $L \in \mathrm{Pic}^\circ(X)$ general, the line bundle $\mathcal{O}_X(p) \otimes L$ is ample but has no sections.

Exercise 2.6. Compute the Picard group of an elliptic curve. Given an example of such a curve admitting a torsion, non-trivial line bundle.

Exercise 2.7. Set $q = \dim H^1(X, \mathcal{O}_X)$. Show that $\mathrm{Pic}^\circ(X)_{\mathrm{tor}} \simeq (\mathbb{Q}/\mathbb{Z})^{2q}$.

Exercise 2.8. Let L be a line bundle such that $c_1(L) = 0$.

1. Show that there exists an hermitian metric h on L such that $\nabla_h^{1,0}$ is a holomorphic connection. Here, ∇_h is the Chern connection of L and being a holomorphic connection means that for any local section $s \in H^0(U, L)$, one has $\nabla_h^{1,0} s \in H^0(U, L \otimes \Omega_X^1)$.
2. Show that L admits local trivializing sections s such that $\nabla_h^{1,0} s = 0$.
3. Show that one can choose transition functions of L that are locally constant. Deduce that the pullback of L to the universal cover of X is trivial.

Exercise 2.9. Let $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ be a representation and let $\pi_1(X)$ acts on $\tilde{X} \times \mathbb{C}^r$ diagonally (via the natural action of $\pi_1(X)$ on $\tilde{X}$ and ρ on $\mathbb{C}^r$. Set $E := (\tilde{X} \times \mathbb{C}^r)/\pi_1(X)$.

1. Show that E admits a structure of holormorphic vector bundle over X.
2. Show that one can choose transition functions for E that are locally constant.
3. In case $r = 1$, show that $c_1(E) = 0$.

Exercise 2.10. Let L be a line bundle such that $c_1(L) = 0$. Denote by $\pi : \tilde{X} \to X$ the universal cover of X.

1. Show that the map $\pi^* : H^1(X, \mathcal{O}_X) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is the zero map (use the isomorphism $H^1(X, \mathcal{O}_X) \simeq H^{0,1}(X)$ and represent classes by harmonic ones).
2. Deduce that $\pi^* L$ is trivial.

Exercise 2.11. Let L be a line bundle on a projective manifold of dimension n. Show that $h^0(X, kL) = O(k^n)$ as $k \to +\infty$. (hint: one can argue by induction and show that $h^0(X, kL) \leqslant k h^0((kL)|_H) + h^0(k(L - H))$).

Exercise 2.12. Let X be a compact Kähler manifold such that $H^{1,0}(X) = 0$. Show (without using the Albanese map) that there is no non-constant map $X \to T$ to a complex torus. (hint: the map $\pi_1(X) \to \pi_1(T)$ factors through $H_1(X, \mathbb{Z})/\mathrm{tor}$).

3 Intersection numbers

Exercise 3.1. Let L be a line bundle such that $c_1(L) = 0$. Show that $H^0(X, L) = 0$ unless L is trivial.

Exercise 3.2. Compute (E^2) where E is the exceptional divisor of the blow up of a point on a surface.

Exercise 3.3. Let $f : X \to C$ be a surjective map with connected fibers from a compact Kähler surface to a compact Riemann surface. Let C° be the complement of the critical locus of f. Given $c_i \in C^\circ$ $(i = 1, 2)$ set $D_i = f^{-1}(c_i)$.

1. Assuming $C = \mathbb{P}^1$, show that $\mathcal{O}_X(D_1)$ is isomorphic to $\mathcal{O}_X(D_2)$.
2. In general, show that $c_1(D_1) = c_1(D_2)$.
3. Show that $c_1(D_1)^2 = 0$.

4 Linear systems

Exercise 4.1. Show that if $\mathbb{B}(L)$ is empty, $\Phi_L(X)$ is not contained in an hyperplane.