Topics in Complex Algebraic Geometry

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Contents

References for this course are [Dem09, Huy05, Laz04, Voi07].

1 Čech cohomology

1.1 Definitions

Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X, and $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ an open covering of X. For the sake of simplicity, we set:

$$U_{\alpha_0 \alpha_1 \ldots \alpha_q} = U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_q}$$

The group $C^q(\mathcal{U}, \mathcal{F})$ of Čech q-cochains is the set of families

$$c = (c_{\alpha_0 \alpha_1 \ldots \alpha_q}) \in \prod_{(\alpha_0, \ldots, \alpha_q) \in I^{q+1}} \mathcal{F}(U_{\alpha_0 \alpha_1 \ldots \alpha_q}).$$

The group structure on $C^q(\mathcal{U}, \mathcal{F})$ is the obvious one deduced from the addition law on sections of $\mathcal{F}$. The Čech differential $\delta^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$ is defined by the formula:

$$(\delta^q c)_{\alpha_0 \ldots \alpha_{q+1}} = \sum_{0 \leqslant j \leqslant q+1} (-1)^j c_{\alpha_0 \ldots \hat{\alpha}_j \ldots \alpha_{q+1}} |_{U_{\alpha_0 \ldots \alpha_{q+1}}} \tag{1.1}$$

and we set $C^q(\mathcal{U}, \mathcal{F}) = 0, \delta^q = 0$ for $q < 0$. In degrees 0 and 1, we get for example:

$$\begin{aligned} q = 0, \quad c = (c_\alpha), \quad (\delta^0 c)_{\alpha\beta} &= c_\beta - c_\alpha \text{ on } U_{\alpha\beta} \\ q = 1, \quad c = (c_{\alpha\beta}), \quad (\delta^1 c)_{\alpha\beta\gamma} &= c_{\beta\gamma} - c_{\alpha\gamma} + c_{\alpha\beta} \text{ on } U_{\alpha\beta\gamma} \end{aligned} \tag{1.2}$$

Easy verifications left to the reader show that $\delta^{q+1} \circ \delta^q = 0$. We get therefore a cochain complex $(C^\bullet(\mathcal{U}, \mathcal{F}), \delta)$, called the complex of Čech cochains relative to the covering $\mathcal{U}$.

Definition 1.3. The Čech cohomology group of $\mathcal{F}$ relative to $\mathcal{U}$ is

$$H^q(\mathcal{U}, \mathcal{F}) = H^q(C^\bullet(\mathcal{U}, \mathcal{F})) = \frac{\operatorname{Ker}(\delta^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F}))}{\operatorname{Im}(\delta^{q-1} : C^{q-1}(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{U}, \mathcal{F}))}.$$

Formula (1.2) shows that the set of Čech 0-cocycles is the set of families $(c_\alpha) \in \prod \mathcal{F}(U_\alpha)$ such that $c_\beta = c_\alpha$ on $U_\alpha \cap U_\beta$. Such a family defines in a unique way a global section $f \in \mathcal{F}(X)$ with $f|_{U_\alpha} = c_\alpha$. Hence:

$$H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X) \tag{1.3}$$

Now, let $\mathcal{V} = (V_\beta)_{\beta \in J}$ be another open covering of X that is finer than $\mathcal{U}$; this means that there exists a map $\rho : J \to I$ such that $V_\beta \subset U_{\rho(\beta)}$ for every $\beta \in J$. Then we can define a morphism $\rho^\bullet : C^\bullet(\mathcal{U}, \mathcal{A}) \to C^\bullet(\mathcal{V}, \mathcal{A})$ by

$$(\rho^q c)_{\beta_0 \ldots \beta_q} = c_{\rho(\beta_0) \ldots \rho(\beta_q)} |_{V_{\beta_0 \ldots \beta_q}} \tag{1.4}$$

the commutation property $\delta\rho^\bullet = \rho^\bullet\delta$ is immediate. If $\rho' : J \to I$ is another refinement map such that $V_\beta \subset U_{\rho'(\beta)}$ for all β, the morphisms $\rho^\bullet, \rho'^\bullet$ are homotopic. To see this, we define a map $h^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q-1}(\mathcal{V}, \mathcal{F})$ by

$$(h^q c)_{\beta_0 \ldots \beta_{q-1}} = \sum_{0 \leqslant j \leqslant q-1} (-1)^j c_{\rho(\beta_0)\ldots\rho(\beta_j)\rho'(\beta_j)\ldots\rho'(\beta_{q-1})}|_{V_{\beta_0\ldots\beta_{q-1}}}.$$

The homotopy identity $\delta^{q-1} \circ h^q + h^{q+1} \circ \delta^q = \rho'^q - \rho^q$ is easy to verify. Hence $\rho^\bullet$ and $\rho'^\bullet$ induce a map depending only on $\mathcal{U}, \mathcal{V}$:

$$H^q(\rho^\bullet) = H^q(\rho'^\bullet) : H^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{V}, \mathcal{F}). \tag{1.5}$$

Now, we want to define a direct limit $H^q(X, \mathcal{F})$ of the groups $H^q(\mathcal{U}, \mathcal{F})$ by means of the refinement mappings (1.5). In order to avoid set theoretic difficulties, the coverings used in this definition will be considered as subsets of the power set $\mathcal{P}(X)$ so that the collection of all coverings becomes actually a set.

Definition 1.1. The Čech cohomology group $H^q(X, \mathcal{F})$ is the direct limit

$$H^q(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{F}) \tag{1.6}$$

when $\mathcal{U}$ runs over the collection of all open coverings of X. Explicitly, this means that the elements of $H^q(X, \mathcal{F})$ are the equivalence classes in the disjoint union of the groups $H^q(\mathcal{U}, \mathcal{F})$, with an element in $H^q(\mathcal{U}, \mathcal{F})$ and another in $H^q(\mathcal{V}, \mathcal{F})$ identified if their images in $H^q(\mathcal{W}, \mathcal{F})$ coincide for some refinement $\mathcal{W}$ of the coverings $\mathcal{U}$ and $\mathcal{V}$.

If $\varphi : \mathcal{F} \to \mathcal{G}$ is a sheaf morphism, we have an obvious induced morphism $\varphi^\bullet : C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{G})$, and therefore we find a morphism

$$H^q(\varphi^\bullet) : H^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{U}, \mathcal{G}).$$

Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of sheaves. We have an exact sequence of groups

$$0 \to C^q(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{U}, \mathcal{G}) \to C^q(\mathcal{U}, \mathcal{H}) \tag{1.7}$$

but in general the last map is not surjective, because every section in $\mathcal{H}(U_{\alpha_0\ldots\alpha_q})$ need not have a lifting in $\mathcal{G}(U_{\alpha_0\ldots\alpha_q})$. The image of $C^\bullet(\mathcal{U}, \mathcal{G})$ in $C^\bullet(\mathcal{U}, \mathcal{H})$ will be denoted $C^\bullet_{\mathcal{G}}(\mathcal{U}, \mathcal{H})$ and called the complex of liftable cochains of $\mathcal{H}$ in $\mathcal{G}$. By construction, the sequence

$$0 \to C^q(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{U}, \mathcal{G}) \to C^q_{\mathcal{G}}(\mathcal{U}, \mathcal{H}) \to 0 \tag{1.8}$$

is exact, thus we get a corresponding long exact sequence of cohomology

$$H^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{U}, \mathcal{G}) \to H^q_{\mathcal{G}}(\mathcal{U}, \mathcal{H}) \to \ldots \tag{1.9}$$

1.2 Čech Cohomology on paracompact spaces

We state here that Čech cohomology theory behaves well on paracompact spaces, namely, we get exact sequences of cohomology for any short exact sequence of sheaves.

Proposition 1.2. *Assume that X is paracompact. If*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is a short exact sequence of sheaves, there is a "long" exact sequence

$$H^q(X,\mathcal{F}) \to H^q(X,\mathcal{G}) \to H^q(X,\mathcal{H}) \to H^{q+1}(X,\mathcal{F}) \to \dots$$

which is the direct limit of the exact sequences (1.9) over all coverings $\mathcal{U}$.

Proof. We have to show that the natural map $\varinjlim H^q_{\mathcal{G}}(\mathcal{U},\mathcal{H}) \longrightarrow \varinjlim H^q(\mathcal{U},\mathcal{H})$ is an isomorphism. This follows from the Lifting Lemma, see e.g. [Dem09, Lemma 5.20]. □

Let us now state the following two useful results which give sufficient conditions for a sheaf to have vanishing cohomology groups (in positive degree).

Proposition 1.3. *Let $\mathcal{F}$ be a sheaf of modules over a sheaf of rings $\mathcal{R}$ on X. Assume that $\mathcal{R}$ is a soft sheaf (i.e. $\mathcal{R}$ admits locally finite partitions of unity for every open covering of X). Then $H^q(\mathcal{U},\mathcal{F}) = 0$ for every $q \geqslant 1$ and every open covering $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ of X.*

Proof. Let $(\psi_\alpha)_{\alpha\in I}$ be a partition of unity in $\mathcal{R}$ subordinate to $\mathcal{U}$, i.e. we have $\psi_\alpha \in \mathcal{R}(X)$ with $\mathrm{Supp}(\psi_\alpha) \subset U_\alpha$ and $\sum_\alpha \psi_\alpha = 1$. We define $h^q : C^q(\mathcal{U},\mathcal{F}) \to C^{q-1}(\mathcal{U},\mathcal{F})$ by

$$(h^q c)_{\alpha_0\dots\alpha_{q-1}} = \sum_{\nu\in I} \psi_\nu c_{\nu\alpha_0\dots\alpha_{q-1}} \tag{1.10}$$

where $\psi_\nu c_{\nu\alpha_0\dots\alpha_{q-1}}$ is extended by 0 on $U_{\alpha_0\dots\alpha_{q-1}} \cap (X \setminus U_\nu)$. It is clear that

$$\begin{aligned}
(\delta^{q-1}h^q c)_{\alpha_0\dots\alpha_q} &= \sum_{j=0}^{q}(-1)^j \sum_{\nu\in I} \psi_\nu c_{\nu\alpha_0\dots\hat{\alpha}_j\dots\alpha_q} \\
&= \sum_{\nu\in I} \psi_\nu(-(\delta^q c)_{\nu\alpha_0\dots\alpha_q} + c_{\alpha_0\dots\alpha_q}) \\
&= (c - h^{q+1}\delta^q c)_{\alpha_0\dots\alpha_q}
\end{aligned}$$

i.e. $\delta^{q-1}h^q + h^{q+1}\delta^q = \mathrm{Id}$. Hence $\delta^q c = 0$ implies $\delta^{q-1}h^q c = c$ if $q \geqslant 1$. □

Proposition 1.4. *Let $\mathcal{F}$ be a sheaf such that for any open set U, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is onto. Then*

$$H^q(X,\mathcal{F}) = 0, \quad \forall q > 0.$$

Proof. It is sufficient to prove that given a cover $\mathcal{U}$ of X and an element $c \in C^q(\mathcal{U},\mathcal{F})$ such that $\delta c = 0$, there exists $b \in C^{q-1}(\mathcal{U},\mathcal{F})$ such that $\delta b = c$. For simplicity, we write the proof for $q = 1$.

We consider

$$S := \{(V, b); V \subset X \,\text{open}, b \in C^0(\mathcal{U} \cap V); \, c_{ij} = b_i - b_j \,\text{on}\, U_i \cap V\},$$

which we endow with the order $(V', b') \leqslant (V, b)$ if $V' \subset V$ and $b' = b|_{V'}$. The proposition is asserting that there exists an element in S of the form (X, b).

First, we observe that the set S is not empty; indeed pick any $U_k \in \mathcal{U}$ and set $V := U_k$, $b_i := c_{ik}$ on U_{ik} if $i \neq k$ and $b_k = 0$ otherwise. Next, by Zorn's lemma, S admits a maximal element $(V_{\max}, b)$. We will prove the proposition by contradiction and assume that $V_{\max} \neq X$.

By assumption, one can find k such that $U_k \not\subset V_{\max}$. The elements b_i are a priori only defined on $U_i \cap V$, and one would like to extend them to elements b'_i on $U_i \cap (V \cup U_k)$ still satisfying the relation $b'_i - b'_j = c_{ij}$ on U_{ijk}. To do so, we observe that the sections

$$\sigma_i := b_i - c_{ik} \in \mathcal{F}(U_{ik} \cap V)$$

coincide with $b_k|_{U_{ik} \cap V}$ hence define a section $\sigma \in \mathcal{F}(U_k \cap V)$. By assumption, there exists an extension $\sigma' \in \mathcal{F}(U_k)$ such that $\sigma'|_{U_{ik} \cap V} = \sigma_i$ for all i. Set

$$b'_i := \begin{cases} b_i & \text{on } U_i \cap V, \\ \sigma' + c_{ik} & \text{on } U_i \cap U_k \ \text{if}\, i \neq k, \\ \sigma' & \text{on } U_k \ \text{if}\, i = k. \end{cases}$$

This is well-defined since on $U_{ik} \cap V$, we have $b_i = \sigma_i + c_{ik} = \sigma' + c_{ik}$. To reach the final contradiction, we are left with checking that $c_{ij} = b'_i - b'_j$ on $U_{ij} \cap (V \cup U_k)$. This is clear on $U_{ij} \cap V$ by construction. And on U_{ijk}, we distinguish three cases. If $i, j \neq k$, then $b'_i - b'_j = (\sigma' + c_{ik}) - (\sigma' + c_{jk}) = c_{ij}$ by the cocycle condition. If $i = k, j \neq k$, then $b'_i - b'_j = \sigma' - (\sigma' + c_{jk}) = -c_{jk} = c_{ij}$. Finally, if $i = j = k$, then $b'_i - b'_j = \sigma' - \sigma' = 0 = c_{kk}$. □

1.3 The De Rham-Weil isomorphism theorem

Let $(\mathcal{L}^\bullet, d)$ be a resolution of a sheaf $\mathcal{F}$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}^0 \xrightarrow{d^0} \mathcal{L}^1 \xrightarrow{d^1} \mathcal{L}^2 \longrightarrow \dots$$

Assume all $\mathcal{L}^q$ are acyclic ($H^s(X, \mathcal{L}^q) = 0$ for $s \geqslant 1$). Set $\mathcal{Z}^q = \ker d^q$. We have $\mathcal{Z}^0 = \mathcal{F}$ and for every $q \geqslant 1$, we get a short exact sequence

$$0 \longrightarrow \mathcal{Z}^{q-1} \longrightarrow \mathcal{L}^{q-1} \xrightarrow{d^{q-1}} \mathcal{Z}^q \longrightarrow 0$$

hence another exact sequence

$$0 \longrightarrow H^s(X, \mathcal{L}^{q-1}) \xrightarrow{d^{q-1}} H^s(X, \mathcal{Z}^q) \xrightarrow{\partial^{s,q}} H^{s+1}(X, \mathcal{Z}^{q-1}) \longrightarrow H^{s+1}(X, \mathcal{L}^{q-1}) = 0.$$

If $s \geqslant 1$, the first group vanishes too and we get an isomorphism

$$\partial^{s,q} : H^s(X, \mathcal{Z}^q) \xrightarrow{\cong} H^{s+1}(X, \mathcal{Z}^{q-1}). \tag{1.11}$$

For $s = 0$, we have $H^0(X, \mathcal{L}^{q-1}) = \mathcal{L}^{q-1}(X)$ and $H^0(X, \mathcal{Z}^q) = \mathcal{Z}^q(X)$, the q-cocycle group of $\mathcal{L}^\bullet(X)$ so that the connecting map $\partial^{0,q}$ induces an isomorphism

$$H^q(\mathcal{L}^\bullet(X)) = \mathcal{Z}^q(X)/d^{q-1}\mathcal{L}^{q-1}(X) \xrightarrow{\tilde{\partial}^{0,q}} H^1(X, \mathcal{Z}^{q-1}).$$

The composite map $\partial^{q-1,1} \circ \ldots \circ \partial^{1,q-1} \circ \tilde{\partial}^{0,q}$ therefore defines an isomorphism

$$H^q(\mathcal{L}^\bullet(X)) \xrightarrow{\tilde{\partial}^{0,q}} H^1(X, \mathcal{Z}^{q-1}) \xrightarrow{\partial^{1,q-1}} \ldots \xrightarrow{\partial^{q-1,1}} H^q(X, \mathcal{Z}^0) = H^q(X, \mathcal{F}).$$

All in all, we have proved

Theorem 1.5 (De Rham-Weil Isomorphism Theorem). *If $(\mathcal{L}^\bullet, d)$ is a resolution of $\mathcal{F}$ by sheaves $\mathcal{L}^q$ which are acyclic on X, there is a functorial isomorphism:*

$$H^q(\mathcal{L}^\bullet(X)) \xrightarrow{\cong} H^q(X, \mathcal{F}).$$

1.4 Application to singular cohomology

As an application of Theorem 1.5, one can prove the following isomorphism

Theorem 1.6. *If X is a manifold, there is an isomorphism*

$$H^q_{\text{sing}}(X, \mathbb{Z}) \simeq H^q(X, \mathbb{Z}_X)$$

where the LHS is the singular cohomology of X with values in $\mathbb{Z}$ and the RHS is the Čech cohomology of locally constant sheaf $\mathbb{Z}_X$.

Proof. Consider the sheaf $\mathcal{C}^q$ associated to the presheaf $U \mapsto C^q_{\text{sing}}(U, \mathbb{Z})$ of singular q-cochains. It satisfies the following properties:

(i) The natural complex

$$0 \to \mathbb{Z}_X \to \mathcal{C}^0 \to \mathcal{C}^1 \to \ldots$$

induced by the coboundary maps is a resolution of $\mathbb{Z}_X$.

(ii) For any open sets $U \subset V$, the restriction map $\mathcal{C}^q(V) \to \mathcal{C}^q(U)$ is surjective.

The first fact is elementary in degree 0, and it follows from $H^q_{\text{sing}}(U, \mathbb{Z}) = 0$ if U is contractible and $q > 0$. As for the second one, any element $\sigma \in C^q_{\text{sing}}(U, \mathbb{Z})$ can be extended by 0 to an element in $C^q_{\text{sing}}(X, \mathbb{Z})$ (i.e. it maps singular chains with supported not included in U to zero). This is easily seen to imply that the associated sheaf $\mathcal{C}^q$ is flasque, too. Combining $(i) - (ii)$ with Proposition 1.4 and Theorem 1.5, we see that

$$H^q(X, \mathbb{Z}) \simeq \mathcal{C}^q(X, \mathbb{Z})/\partial \mathcal{C}^{q-1}(X, \mathbb{Z}). \tag{1.12}$$

Next, it is not difficult to see that we have an isomorphism

$$\mathcal{C}^q(X, \mathbb{Z}) \simeq C^q_{\text{sing}}(X, \mathbb{Z})/C^q_{\text{sing}}(X, \mathbb{Z})_0 \tag{1.13}$$

where $C^q_{\text{sing}}(X,\mathbb{Z})_0$ consists of cochains which vanish in restriction to (small) open sets U_i which form a covering of X. E.g. let us explain why we the natural map

$$\Phi : C^q_{\text{sing}}(X,\mathbb{Z}) \to \mathcal{C}^q(X,\mathbb{Z})$$

is surjective. If $\beta \in \mathcal{C}^q(X,\mathbb{Z})$, it can be represented by $\beta_i \in C^q(U_i,\mathbb{Z})$ for (U_i) a covering of X by small enough open sets, with the natural compatibility conditions. We now define a singular cochain $\tilde{\beta} \in C^q(X,\mathbb{Z})$ as follows. Let $\sigma \in C_{q,\text{sing}}(X,\mathbb{Z})$; if there exists i such that $\text{Supp}(\sigma) \subset U_i$, then we set $\tilde{\beta}(\sigma) = \beta_i(\sigma)$. Otherwise we decree that $\tilde{\beta}(\sigma) = 0$. It is elementary to check that $\tilde{\beta}$ is well-defined and that $\Phi(\tilde{\beta}) = \beta$, as the latter equality can be checked locally given that $\mathcal{C}^q$ is a sheaf.

Recall that the theorem of small chains [Spa82, § 4.4] says that the complex of chains with support in a given small covering of X computes the singular homology of X. In other words, the cohomology of the complex

$$C^\bullet_{\text{sing}}(X,\mathbb{Z})/C^\bullet_{\text{sing}}(X,\mathbb{Z})_0$$

coincides with that of the usual complex $C^\bullet_{\text{sing}}(X,\mathbb{Z})$. The theorem now follows from (1.12) and (1.13). □

Remark 1.7. As we saw in the proof above, a good deal of the difficulty in the proof above stems from the fact that $U \mapsto C^q_{\text{sing}}(U,\mathbb{Z})$ is not a sheaf. Here's a simple example that illustrates that fact. Take $X = S^1$ and cover $X = U_1 \cup U_2$ by contractible open sets. We define $\beta \in C^1_{\text{sing}}(X,\mathbb{Z})$ by decreeing $\beta(\sigma) = 1$ if $\text{Im}(\sigma) = S^1$ and $\beta(\sigma) = 0$ otherwise. Clearly, we have $\beta \neq 0$ but $\beta|_{U_i} = 0$ for $i = 1,2$.

2 Cohomology of a compact Kähler manifold

2.1 The de Rham cohomology

Given a differentiable manifold of dimension n and given an integer $0 \leqslant k \leqslant n$, we will denote by $\mathcal{A}^k_M := \Lambda^k T^*_M$ the differentiable vector bundle of smooth k-forms. If the base manifold is fixed, we will sometimes write $\mathcal{A}^k$ instead of $\mathcal{A}^k_M$.

Let us start by recalling the well-known Poincaré lemma on the unit ball in the euclidean space.

Lemma 2.1 (Poincaré lemma)**.** *Let $\alpha \in \mathcal{C}^\infty(B,\mathcal{A}^k)$ be a k-form on the unit ball $B \subset \mathbb{R}^m$ with $k > 0$. If $d\alpha = 0$, then there exists a $(k-1)$-form $\beta \in \mathcal{C}^\infty(B,\mathcal{A}^{k-1})$ such that $\alpha = d\beta$.*

Let now M be a differentiable real manifold of dimension m, let $\mathcal{A}^k_M$ be the $\mathcal{C}^\infty$ vector bundle of differential k forms, and let d be the exterior differential. Closed forms are in general not exact (take $\alpha = d\theta$ on $M = S^1$) and the de Rham cohomology spaces measure this obstruction.

Definition 2.2 (de Rham cohomology)**.** Let M be a differentiable manifold of dimension m. For any integer $0 \leqslant k \leqslant n$, one defines

$$H^k(M,\mathbb{R}) := \{\alpha \in \mathcal{C}^\infty(M,\mathcal{A}^k_M);\ d\alpha = 0\} \Big/ \{d\beta; \beta \in \mathcal{C}^\infty(M,\mathcal{A}^{k-1}_M)\}\ .$$

Remark 2.3 (Comparison with singular cohomology)**.** Consider the following complex

$$\mathbb{R}_M \hookrightarrow \mathcal{A}^0_M \xrightarrow{d} \mathcal{A}^1_M \xrightarrow{d} \mathcal{A}^2_M \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^m_M \xrightarrow{d} 0$$

where $\mathbb{R}_M$ is the locally constant sheaf on M with values in $\mathbb{R}$, which is naturally a subsheaf of the sheaf of *smooth* 0-forms (i.e. functions). The Poincaré lemma says that this complex is actually a resolution of $\mathbb{R}_M$. Since the sheaves $\mathcal{A}^k_M$ are soft as $\mathcal{C}^\infty_M$-modules, they are acyclic (i.e. they have no cohomology), cf Proposition 1.3. De Rham-Weil isomorphism theorem (Theorem 1.5) shows that we have an isomorphism

$$H^k(M,\mathbb{R}) \simeq \check{H}^k(M,\mathbb{R}_M) \tag{2.14}$$

where the RHS is the Čech cohomology in degree k of $\mathbb{R}_M$, itself isomorphic to the singular cohomology of M with values in $\mathbb{R}$ thanks to Theorem 1.6.

Let us give below explicit construction of that isomorphism in degree 2.

Lemma 2.4. *There is an explicit isomorphism*

$$H^2(M,\mathbb{R}) \simeq \check{H}^2(M,\mathbb{R}_M) \tag{2.15}$$

Proof. Let $[\omega] \in H^2(X,\mathbb{R})$, such that $\omega|_{U_\alpha} = dA_\alpha$ for some 1-form A_α and a suitable cover $X = \cup_\alpha U_\alpha$. We get an element $(A_\alpha - A_\beta) \in H^1(X,\mathcal{Z}^1)$, and one writes $A_\alpha - A_\beta = df_{\alpha\beta}$ on $U_{\alpha\beta}$ for some functions $f_{\alpha\beta}$. This defines a 2-cocycle $f = (f_{\alpha\beta\gamma})$ with values in $\mathbb{R}$ by $f_{\alpha\beta\gamma} := f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma}$. We set $\Phi([\omega]) := [f]$, and one checks immediately that Φ is well-defined.

Conversely, let $[c] \in \check{H}^2(X,\mathbb{R}_M)$ be represented by a 2-cocycle $c = (c_{\alpha\beta\gamma})$ with constant coefficients. Let (χ_ν) be a partition of unity subordinate to our cover. We define $b_{\alpha\beta} = d(\sum_\nu \chi_\nu c_{\nu\alpha\beta})$ which is a 1-cocycle with values in $\mathcal{A}^1$. Indeed, $b_{\beta\gamma} - b_{\alpha\gamma} + b_{\alpha\beta} = d(\sum_\nu c_{\alpha\beta\gamma}) = dc_{\alpha\beta\gamma} = 0$ since $\delta c = 0$ and $c_{\alpha\beta\gamma}$ is constant. Next, define $a = (a_\alpha)$ with $a_\alpha := d(\sum_\nu \chi_\nu b_{\nu\alpha})$ which is a 0-cocycle with values in locally exact 2-forms. In other words, a is well-defined closed 2-form. We set $\Psi([c]) = [a]$ which again is easily seen to be a well-defined map.

It is not hard to see that the two constructions are inverse of each other. E.g. start with $c = (c_{\alpha\beta\gamma}) \in \check{H}^2(X,\mathbb{R}_M)$ as above, with associated 2 form $a = (a_\alpha)$ such that $a_\alpha = d(\sum_\nu \chi_\nu b_{\nu\alpha})$ and $b_{\alpha\beta} = d(\sum_\nu \chi_\nu c_{\nu\alpha\beta})$. The 2-cocycle $f = (f_{\alpha\beta\gamma})$ associated to a is obtained as follows. First, write $a_\alpha - a_\beta = -b_{\alpha\beta} = -df_{\alpha\beta}$ with $f_{\alpha\beta} = \sum_\nu \chi_\nu c_{\nu\alpha\beta}$. This implies that $f_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma} = c_{\alpha\beta\gamma}$, i.e. $f = c$. In particular, we have $\Phi \circ \Psi = -\mathrm{Id}$.

In the other direction, start with $[\omega] \in H^2(X,\mathbb{R})$ and consider $A_\alpha, f_{\alpha\beta}, f_{\alpha\beta\gamma}$ as before. The class $\Psi \circ \Phi([\omega])$ is represented on U_α by

$$\begin{aligned} d\Big[\sum_\nu \chi_\nu d\big[\sum_\lambda \chi_\lambda f_{\lambda\nu\alpha}\big]\Big] &= d\Big[\sum_\nu \chi_\nu d\big[\sum_\lambda \chi_\lambda (f_{\lambda\nu} + f_{\nu\alpha} - f_{\lambda\alpha})\big]\Big] \\ &= d\Big[\sum_\nu \chi_\nu d\big[\sum_\lambda \chi_\lambda f_{\lambda\nu}\big]\Big] + d\big(\sum_\nu \chi_\nu A_\nu\big) - dA_\alpha \\ &\quad -d\Big[\sum_\nu \chi_\nu d\big[\sum_\lambda \chi_\lambda f_{\lambda\alpha}\big]\Big] \end{aligned}$$

The first two terms are globally defined exact forms, the third term is $-\omega$. As for the last one, it is equal to $-d^2(\sum_\lambda \chi_\lambda f_{\lambda\alpha}) = 0$. All in all, we have found $\Psi \circ \Phi = -\mathrm{Id}$. $\square$

Definition 2.5 (Cup product)**.** The total de Rham cohomology

$$H^\bullet(M,\mathbb{R}) = \bigoplus_{k=0}^{m} H^k(M,\mathbb{R})$$

has a natural ring structure provided by the cup product, which is defined by the wedge product at the level of forms. More precisely, if α, β are two closed forms of degree k and ℓ respectively, then one sets $[\alpha] \cup [\beta] := [\alpha \wedge \beta] \in H^{k+\ell}(M,\mathbb{R})$; which is well-defined thanks to Leibniz formula.

Recall that a connected differentiable manifold M of dimension m is called orientable if there exists a non-vanishing top form $\omega \in \mathcal{C}^\infty(M, \mathcal{A}^m_M)$; that is a trivialization of the line bundle $\mathcal{A}^m_M$. An orientation is a choice of one such form. If M is compact then Stokes formula shows that a trivialization ω is never exact, hence $H^m(M,\mathbb{R}) \neq 0$. More precisely, we have the following duality theorem

Theorem 2.6 (Poincaré duality)**.** *Let M be a connected, orientable differentiable manifold of dimension m and let $0 \leqslant k \leqslant m$ be an integer. Then, the pairing*

$$\begin{aligned} H^k(M,\mathbb{R}) \times H^{m-k}(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ ([\alpha],[\beta]) &\longmapsto \int_M \alpha \wedge \beta \end{aligned}$$

is non-degenerate.

In particular, the integration yields isomorphism $H^m(M,\mathbb{R}) \simeq \mathbb{R}$.

2.2 Complex-valued differential forms

Let X be a complex manifold of dimension n. We denote by T_X (resp. Ω_X) the holomorphic tangent bundle (resp. holomorphic cotangent bundle) of X.

Definition 2.7 (The complex vector bundles $T^{1,0}_X$ and $T^{0,1}_X$)**.** The real tangent bundle $T_{X,\mathbb{R}}$, i.e. the tangent bundle of X seen as a real differentiable manifold, is a $\mathcal{C}^\infty$ real vector bundle with rank $2n$ endowed with an action $J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ induced by the multiplication by i on T_X; it satisfies $J^2 = -\mathrm{Id}_{T_{X,\mathbb{R}}}$. We decompose the complexified tangent bundle $T_{X,\mathbb{C}} := T_{X,\mathbb{R}} \otimes \mathbb{C}$

$$T_{X,\mathbb{C}} = T^{1,0}_X \oplus T^{0,1}_X \tag{2.16}$$

according to its eigenspaces, where $T^{1,0}_{X,x} := \{u \in (T_{X,\mathbb{C}})_x ; Ju = iu\}$ and $T^{0,1}_{X,x} := \{u \in (T_{X,\mathbb{C}})_x ; Ju = -iu\}$ is the complex conjugate of $T^{1,0}_{X,x}$.

The canonical realization of the *holomorphic* tangent bundle T_X inside $T_{X,\mathbb{R}} \otimes \mathbb{C}$ is isomorphic to $T^{1,0}_X$ (as $\mathcal{C}^\infty$ complex vector bundles).

Local picture. Under a local trivialization of $X \supset U \simeq \mathbb{C}^n$, we have complex coordinates $z_1, \ldots, z_n$ as well as real coordinates $x_k = \mathrm{Re}(z_k), y_k = \mathrm{Im}(z_k)$ which induce vector fields $\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k} \in T_{X,\mathbb{R}}$ and $\frac{\partial}{\partial z_k} \in T_X$. Then $T_{X,\mathbb{R}} := \oplus_{k=1}^n (\mathbb{R}\frac{\partial}{\partial x_k} \oplus \mathbb{R}\frac{\partial}{\partial y_k})$ and the operator J satisfies $J\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}$ and $J\frac{\partial}{\partial y_k} = -\frac{\partial}{\partial x_k}$. Since $\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)$, we find $J\frac{\partial}{\partial z_k} = i\frac{\partial}{\partial z_k}$. The conjugation operator $T_{X,\mathbb{R}} \otimes \mathbb{C} \ni v \otimes \lambda \mapsto v \otimes \bar{\lambda}$ is a $\mathbb{C}$-antilinear automorphism such that $\overline{\frac{\partial}{\partial z_k}} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) =: \frac{\partial}{\partial \bar{z}_k}$ satisfies $J\frac{\partial}{\partial \bar{z}_k} = -i\frac{\partial}{\partial \bar{z}_k}$. In particular, we find that over U, $T_X^{1,0} = \oplus_{k=1}^n \mathbb{C}\frac{\partial}{\partial z_k}$ and $T_X^{0,1} = \oplus_{k=1}^n \mathbb{C}\frac{\partial}{\partial \bar{z}_k}$, and finally, $\overline{T_X^{1,0}} = T_X^{0,1}$.

We define the complex vector bundles $\mathcal{A}_{X,\mathbb{C}} := T^*_{X,\mathbb{C}}, \mathcal{A}_X^{1,0} := (T_X^{1,0})^*$, and similarly $\mathcal{A}_X^{0,1} := (T_X^{0,1})^*$. Dualizing (2.16), one gets

$$\mathcal{A}_{X,\mathbb{C}} = \mathcal{A}_X^{1,0} \oplus \mathcal{A}_X^{0,1} \tag{2.17}$$

and similarly as for the tangent, we see that $\mathcal{A}_X^{1,0}$ is spanned by the 1-forms $dz_k := \frac{1}{2}(dx_k + idy_k)$ and $\Omega_X^{0,1}$ is spanned by the 1-forms $d\bar{z}_k := \frac{1}{2}(dx_k - idy_k)$.

Finally, one defines for any integer $1 \leqslant k \leqslant n$ the following complex vector bundle $\mathcal{A}^k_{X,\mathbb{C}} := \Lambda^k \mathcal{A}_{X,\mathbb{C}}$ as well as for any $1 \leqslant p, q \leqslant n$, $\mathcal{A}_X^{p,q} := \Lambda^p \mathcal{A}_X^{1,0} \otimes \Lambda^q \mathcal{A}_X^{0,1}$. One gets a decomposition

$$\mathcal{A}^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}_X^{p,q}. \tag{2.18}$$

Locally, a smooth (p,q)-form α can be uniquely represented as $\alpha = \sum_{I,J} f_{IJ} dz_I \wedge d\bar{z}_J$ where the sum runs over all ordered subsets $I, J \subset \{1, \ldots, n\}$ with p (resp. q) elements, and f_{IJ} are smooth, complex-valued functions. Moreover, if $I = \{i_1, \ldots, i_p\}$, one writes $dz_I := dz_{i_1} \wedge \ldots \wedge dz_{i_p}$ and similarly for $d\bar{z}_J$.

Definition 2.8 (Real forms)**.** A form $\alpha \in \mathcal{C}^\infty(X, \mathcal{A}_X^k)$ is real if $\alpha = \bar{\alpha}$. If, moreover, $\alpha \in \mathcal{C}^\infty(X, \mathcal{A}_X^{p,q})$, then $p = q$ unless $\alpha = 0$. In local coordinates, a (p,p)-form $\alpha = \sum_{I,J} f_{IJ} dz_I \wedge d\bar{z}_J$ is real if and only if $f_{IJ} = -\overline{f_{JI}}$.

Definition 2.9 (Kähler metrics, Kähler manifolds)**.** Let X be a complex manifold. A hermitian metric on X is a hermitian positive definite form of class $\mathcal{C}^\infty$ on T_X. In local coordinates, $h = \sum_{k,\ell} h_{k\ell} dz_k \otimes d\bar{z}_\ell$. One associate to h the fundamental form $\omega := -\mathrm{Im}(h)$. In local coordinates, $\omega = \frac{i}{2}\sum_{k\ell} h_{k\ell} dz_k \wedge d\bar{z}_\ell$. It is real, of type $(1,1)$. On says that ω is Kähler if $d\omega = 0$. Finally, X is a Kähler manifold if it admits a Kähler metric.

2.3 The Dolbeault cohomology

The exterior differential $d : \mathcal{A}^k_{X,\mathbb{R}} \to \mathcal{A}^{k+1}_{X,\mathbb{R}}$ extends by $\mathbb{C}$-linearity to $d : \mathcal{A}^k_{X,\mathbb{C}} \to \mathcal{A}^{k+1}_{X,\mathbb{C}}$. Moreover, if α is a smooth (p,q)-form, it is clear from Leibniz rule that one can uniquely decompose $d\alpha = (d\alpha)_{p+1,q} + (d\alpha)_{p,q+1}$ according to its type. We define $\partial\alpha$ (resp. $\bar{\partial}\alpha$) to be the $(p+1,q)$-component (resp. $(p,q+1)$) of $d\alpha$. In local coordinates, we get the following expression for ∂ and $\bar\partial$:

$$\partial\alpha = \sum_{I,J}\sum_{k=1}^{n} \frac{\partial f_{IJ}}{\partial z_k} dz_k \wedge dz_I \wedge d\bar z_J$$

$$\bar\partial\alpha = \sum_{I,J}\sum_{k=1}^{n} \frac{\partial f_{IJ}}{\partial \bar z_k} d\bar z_k \wedge dz_I \wedge d\bar z_J$$

In particular, a $(p,0)$-form $\alpha \in \mathcal{C}^\infty(X, \mathcal{A}_{X,\mathbb{C}}^{p,0})$ is holomorphic if and only if $\bar\partial\alpha = 0$.

Clearly, we have $d = \partial + \bar\partial$. Since $d^2 = 0$, we infer

$$\partial^2 = \bar\partial^2 = 0 \quad \text{and} \quad \partial\bar\partial = -\bar\partial\partial.$$

In particular, a $\bar\partial$-exact form is $\bar\partial$-closed (and the same holds with ∂). While the converse is certainly not true (take $\alpha = \frac{d\bar z}{\bar z}$ on $\mathbb{C}^*$), it is true *locally*. More precisely, we have the complex analogue of Poincaré lemma, cf [Voi07, Proposition 2.31].

Lemma 2.10 (Dolbeault-Grothendieck lemma)**.** *Let α be a (p,q)-form on the unit polydisk $D \subset \mathbb{C}^n$ with $q > 0$. If $\bar\partial\alpha = 0$, then there exists a $(p,q-1)$-form β on D such that $\alpha = \bar\partial\beta$.*

Definition 2.11 (Dolbeault cohomology)**.** Let X be a complex manifold of dimension n. For any indices $0 \leqslant p,q \leqslant n$, one defines

$$H^{p,q}(X) := \{\alpha \in \mathcal{C}^\infty(X, \mathcal{A}_X^{p,q});\ \bar\partial\alpha = 0\} \Big/ \{\bar\partial\beta; \beta \in \mathcal{C}^\infty(X, \mathcal{A}_X^{p,q-1})\}\ .$$

At this point, the $\mathbb{C}$-vector spaces $H^{p,q}(X)$ could be infinite dimensional. For instance, $H^{0,0}(X) = \mathcal{O}_X(X)$ is the space of holomorphic functions on X.

Remark 2.12 (Comparison with coherent cohomology)**.** Consider the following complex

$$\Omega_X^p \hookrightarrow \mathcal{A}_X^{p,0} \xrightarrow{\bar\partial} \mathcal{A}_X^{p,1} \xrightarrow{\bar\partial} \mathcal{A}_X^{p,2} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{A}_X^{p,n} \xrightarrow{\bar\partial} 0$$

where Ω_X^p is the sheaf of *holomorphic* p-forms, which is naturally a subsheaf of the sheaf of *smooth* $(p,0)$-forms. The Dolbeault-Grothendieck lemma says that this complex is actually a resolution of Ω_X^p. Since the sheaves $\mathcal{A}_X^{p,q}$ are soft (as $\mathcal{C}_X^\infty$-modules), the De Rham-Weil isomorphism theorem (Theorem 1.5) shows that we have an isomorphism

$$H^{p,q}(X) \simeq H^q(X, \Omega_X^p) \tag{2.19}$$

where the RHS is the coherent cohomology in degree q of Ω_X^p (or, equivalenty, the Čech cohomology of that sheaf).

2.4 The Hodge decomposition theorem

When (X,g) is a *compact* Riemannian manifold, then Hodge theory shows that the de Rham cohomology

$$H^k(X,\mathbb{R}) := \{\alpha \in \mathcal{C}^\infty(X, \mathcal{A}_X^k);\ d\alpha = 0\} \Big/ \{d\beta; \beta \in \mathcal{C}^\infty(X, \mathcal{A}_X^k)\}$$

is isomorphic to the space $\mathcal{H}^k_g(X)$ of harmonic k-forms (i.e. k-forms α such that $\Delta_d\alpha = 0$, where $\Delta_d = dd^* + d^*d$ is the Hodge laplacian associated to g). In other words, each de Rham cohomology class contains a unique harmonic representative. A first consequence is that $H^k(X)$ is finite dimensional. As another application, one can use this correspondence to prove the Poincaré duality, cf [Voi07, Theorem 5.30]

Let us now assume that X admits a complex structure J such that $J \in O(g)$ and $\omega := g(\cdot, J\cdot)$ is closed; hence ω is a Kähler form. We can consider several laplacians, among which $\Delta_d = dd^* + d^*d$, $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ and $\Delta_\partial = \partial\partial^* + \partial^*\partial$. The last two laplacians have the advantage of preserving the type, i.e. if $\alpha \in \mathcal{C}^\infty(X, \mathcal{A}^{p,q}_X)$, then $\Delta_{\bar\partial}\alpha \in \mathcal{C}^\infty(X, \mathcal{A}^{p,q}_X)$. In particular, usual Hodge theory shows that we have an isomorphism between $H^{p,q}(X)$ and the space $\mathcal{H}^{p,q}_{\bar\partial}(X)$ of $\Delta_{\bar\partial}$-harmonic (p,q)-forms.

One of the main results in Hodge theory states that when ω is Kähler, then we have the following identity between the real and complex laplacian $\frac{1}{2}\Delta_d = \Delta_{\bar\partial} = \Delta_\partial$. In particular, we have $\overline{\Delta_{\bar\partial}} = \Delta_\partial = \Delta_{\bar\partial}$ hence $H^{p,q}_{\bar\partial}(X) \simeq \overline{H^{q,p}_{\bar\partial}(X)}$. Moreover if α is a complex-valued Δ_d-harmonic k-form and $\alpha = \sum_{p+q=k}\alpha_{p,q}$ is its type decomposition, then $\alpha_{p,q}$ is $\Delta_{\bar\partial}$-harmonic as well. This yields the following foundational result

Theorem 2.13 (Hodge decomposition theorem)**.** *Let X be a compact Kähler manifold.There are isomorphisms*

$$H^k(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X)$$

$$H^{q,p}(X) \simeq \overline{H^{q,p}(X)}.$$

One can even refine the statement making the isomorphisms canonical (i.e. independent of the choice of a Kähler metric).

One should insist on the fact that the decomposition above in cohomology does not arise directly from the decomposition $\mathcal{A}^k_{X,\mathbb{C}} = \oplus_{p+q=}\mathcal{A}^{p,q}_X$. For instance, consider $\alpha = \bar z_2 dz_1 + z_1 d\bar z_2$. Then α is closed ($d\alpha = 0$) but its components are not $\bar\partial$-closed ($\bar\partial(\bar z_2 dz_1) = -dz_1 \wedge d\bar z_2 \neq 0$).

For $p = q$, $H^{p,p}(X) \subset H^{2p}(X, \mathbb{C})$ is a complex vector space which is stable under conjugation (recall that $H^{2p}(X, \mathbb{C}) = H^{2p}(X, \mathbb{R}) \otimes_\mathbb{R} \mathbb{C}$ by the universal coefficient theorem). In particular, there is a real vector space $H^{p,p}(X, \mathbb{R}) \subset H^{p,p}(X)$ such that $H^{p,p}(X, \mathbb{R}) \otimes_\mathbb{R} \mathbb{C}$. One can identify $H^{p,p}(X, \mathbb{R})$ with $H^{p,p}(X) \cap H^{2p}(X, \mathbb{R})$, i.e. it consists of (p,p)-classes that can be represented by real, $\bar\partial$-closed (p,p)-forms.

3 Line bundles and divisors

3.1 Basic definitions

Let X be a complex manifold of dimension n.

Definition 3.1 (Line bundle)**.** A line bundle L on X consists of a complex manifold L endowed with a projection map $p : L \to X$ such that X admits an open covering $X = \cup U_\alpha$

and isomorphisms

$$\begin{tikzcd} L|_{p^{-1}(U_\alpha)} \arrow[rr, "\tau_\alpha", "\simeq"'] \arrow[dr, "p"'] & & U_\alpha \times \mathbb{C} \arrow[dl, "\mathrm{pr}_2"] \\ & U_\alpha & \end{tikzcd}$$

such that on the overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$, the isomorphism $\tau_\alpha \circ \tau_\beta^{-1} : U_{\alpha\beta} \times \mathbb{C} \to U_{\alpha\beta} \times \mathbb{C}$ is given by $(x, v) \mapsto (x, g_{\alpha\beta}(x)v)$ for some $g_{\alpha\beta} \in \mathcal{O}_X(U_{\alpha\beta})^*$. In particular, one has $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_{\alpha\beta\gamma}$.

If $x \in X$, we denote by $L_x := p^{-1}(x)$ the fiber of p at x; which is non-canonically isomorphic to $\mathbb{C}$. Up to refining the cover, one can assume that the sets U_α are isomorphic to a polydisk in $\mathbb{C}^n$, and that the double overlaps $U_{\alpha\beta}$ are simply connected.

Given two line bundles L, L' over X, one can consider their tensor product $L \otimes L'$. This is done fiberwise and yields a well-defined line bundle on X. If $X = \cup U_\alpha$ is a covering of X trivializing both L and L' (with respective transition functions $g_{\alpha\beta}$ and $g'_{\alpha\beta}$), then $L \otimes L'$ is also trivialized with transition functions $g_{\alpha\beta}g'_{\alpha\beta}$.

A morphism of line bundles $\phi : L \to L'$ over X is a holomorphic map such that $p' \circ \phi = p$ and such that for any $x \in X$, the induced map $\phi_x : L_x \to L'_x$ is linear (i.e. it is an homothety). It is an isomorphism if it admits an inverse. The (abelian) group of line bundles modulo isomorphism is denoted by $\mathrm{Pic}(X)$.

Proposition 3.2. *There is an isomorphism*

$$\mathrm{Pic}(X) \longrightarrow H^1(X, \mathcal{O}_X^*).$$

Proof. The map is defined by sending L to $(g_{\alpha\beta})$ with the notation above. One can construct an inverse as follows. Given a cocycle $(g_{\alpha\beta})$, we define a sheaf L by $L(U) = \{(f_\alpha); f_\alpha \in \mathcal{O}(U \cap U_\alpha), f_\alpha|_{U_{\alpha\beta}\cap U} = g_{\alpha\beta} f_\beta|_{U_{\alpha\beta}\cap U}\}$. On each U_α, the element e^α defined by $e^\alpha_\beta = g_{\beta\alpha}$ if $\beta \neq \alpha$, $e^\alpha_\alpha = 1$ defines a section of L on U_α. It is elementary to check that if $f \in L(U_\alpha)$, then $f = f_\alpha e^\alpha$ so that L is indeed a line bundle.

If we change $g_{\alpha\beta}$ into $g'_{\alpha\beta} = \frac{g_\alpha}{g_\beta} g_{\alpha\beta}$, we get a new sheaf L' and the maps $g_\alpha : L|_{U_\alpha} \to L'|_{U_\alpha}$ glue to an isomorphism $L \to L'$.

Conversely, if $\Phi : L \to L'$ is an isomorphism, consider local trivialization e_α (resp. e'_α) of $L|_{U_\alpha}$ (resp. $L'|_{U_\alpha}$. There exists holomorphic non-vanishing functions g_α such that $\Phi(e_\alpha) = g_\alpha e'_\alpha$. Since $e_\beta = g_{\alpha\beta} e_\alpha$ (and likewise for L'), we find $g'_{\alpha\beta} = \frac{g_\alpha}{g_\beta} g_{\alpha\beta}$, hence the cocycles $(g_{\alpha\beta})$ and $(g'_{\alpha\beta})$ differ by an exact cocycle. □

Definition 3.3 (Sections)**.** If $U \subset X$, a smooth (resp. holomorphic) section of L over U is a smooth (resp. holomorphic) map $s : U \to L$ such that $p(s(x)) = x$ for all $x \in U$. That is, $s(x) \in L_x$ for all $x \in U$. We write $s \in \mathcal{C}^\infty(U, L)$ (resp. $s \in H^0(U, L)$).

If s is a section of L over X, then $\tau_\alpha(s(x)) = (x, \sigma_\alpha(x))$ for some *function* $\sigma_\alpha : U_\alpha \to \mathbb{C}$. On the overlap $U_{\alpha\beta}$, one has $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$. Conversely, the data of functions $\sigma_\alpha : U_\alpha \to \mathbb{C}$ satisfying the relation $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$ on the overlaps induces a unique section s of L corresponding to the σ_α under the trivialization.

Definition 3.4 (Meromorphic sections)**.** A meromorphic section of a line bundle L on X consists of the data of meromorphic functions $\sigma_\alpha \in \mathcal{M}_X(U_\alpha)$ such that $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$ on every overlap $U_{\alpha\beta}$.

Let us add a few of remarks:

• Global holomorphic sections $s \in H^0(X, L)$ may not exist, although $\mathcal{C}^\infty(X, L)$ is an infinite dimensional vector space, since once can construct many smooth sections using partitions of unity or even just cut-off functions.

• Locally, the map τ_α induces a non-vanishing, holomorphic section $x \mapsto e_\alpha(x) := \tau_\alpha^{-1}(x, 1)$ called local trivialization of L on U_α.

• If s, s' are two meromorphic sections of L, then the quotient $\frac{s}{s'}$ defines a meromorphic *function* on X (provided $s' \not\equiv 0$).

3.2 First Chern class of a line bundle : two approaches

3.2.1 The cocycle point of view

The data of $(U_{\alpha\beta}, g_{\alpha\beta})$ characterizes L up to isomorphism of line bundles. Moreover, one has the following relation

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on } U_{\alpha\beta\gamma}.$$

In particular, L defines a unique cocycle in $H^1(X, \mathcal{O}_X^*)$ and the map that sends L to that cocycle induces a group isomorphism

$$\mathrm{Pic}(X) \xrightarrow{\simeq} H^1(X, \mathcal{O}_X^*),$$

cf Proposition 3.2.

Now, the exponential exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow \mathcal{O}_X \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}_X^* \longrightarrow 0$$

induces a long exact sequence in cohomology and, in particular, a map

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{\psi} \check{H}^2(X, \underline{\mathbb{Z}}_X)$$

that can be described as follows. If $(U_{\alpha\beta}, g_{\alpha\beta})_{\alpha\beta}$ is a cocycle with values in $\mathcal{O}_X^*$, and since $U_{\alpha\beta}$ is simply connected, there exists $f_{\alpha\beta} \in \mathcal{O}_X(U_{\alpha\beta})$ satisfying

$$e^{2\pi i f_{\alpha\beta}} = g_{\alpha\beta}. \tag{3.20}$$

On the triple overlap $U_{\alpha\beta\gamma}$, we have $c_{\alpha\beta\gamma} := f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma} \in \underline{\mathbb{Z}}_X(U_{\alpha\beta\gamma})$, and $(U_{\alpha\beta\gamma}, c_{\alpha\beta\gamma})_{\alpha\beta\gamma}$ defines an element in $H^2(X, \underline{\mathbb{Z}}_X)$. Finally, the inclusion map $\underline{\mathbb{Z}}_X \hookrightarrow \underline{\mathbb{R}}_X$ induces a map $j : \check{H}^2(X, \underline{\mathbb{Z}}_X) \to \check{H}^2(X, \underline{\mathbb{R}}_X)$. One defines the first Chern class of a line bundle L as the image of the composite map $c_1 := j \circ \psi$

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[r, "\psi"] \arrow[rr, bend left, "c_1"] & \check{H}^2(X, \underline{\mathbb{Z}}) \arrow[r, "j"] & \check{H}^2(X, \underline{\mathbb{R}}) \end{tikzcd} \tag{3.21}$$

where one has used the identification $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*)$.

3.2.2 The metric point of view

Definition 3.5 (Hermitian metric)**.** Let $L \to X$ be a line bundle on a complex manifold X. A hermitian metric h on L is a collection of hermitian metrics $(h_x)_{x \in X}$ on L_x that vary smoothly with x.

Concretely, h can be viewed in the trivializations $L|_{p^{-1}(U_\alpha)} \simeq U_\alpha \times \mathbb{C}$ as a map $(x, \lambda) \mapsto |\lambda|^2 e^{-\phi_\alpha}$ where $\phi_\alpha : U_\alpha \to \mathbb{R}_{>0}$ is a smooth function. The choice to write a positive function as an exponential is of course arbitrary, but we will see later that it is particularly convenient. Alternatively, one can use the trivialization $e_\alpha \in H^0(U_\alpha, L)$ to describe h in the chart U_α by setting $|e_\alpha|_h^2 := e^{-\phi_\alpha}$. The functions $\phi_\alpha \in \mathscr{C}^\infty(U_\alpha, \mathbb{R})$ are called local weights of h; of course they depend on the choice of local trivializations of L.

The functions ϕ_α do not match on the overlap $U_{\alpha\beta}$ in general, but since $e_\beta = g_{\alpha\beta} e_\alpha$, we find the relation $\phi_\alpha - \phi_\beta = \log |g_{\alpha\beta}|^2$. Since $g_{\alpha\beta}$ is holomorphic and non-vanishing on $U_{\alpha\beta}$, we have $\partial\bar\partial\phi_\alpha = \partial\bar\partial\phi_\beta$ on $U_{\alpha\beta}$.

Conversely, if L is given, one can construct a hermitian metric h on L by setting $|e_\alpha|_h^2 := \sum_\gamma \chi_\gamma |g_{\gamma\alpha}|^2$ where χ_γ is a partition of unity subordinate to $(U_\gamma)_\gamma$. In other words, one set

$$\phi_\alpha := -\log \sum_\gamma \chi_\gamma |g_{\gamma\alpha}|^2. \tag{3.22}$$

Given the relation $e_\beta = g_{\alpha\beta} e_\alpha$, h is well-defined if and only if we have $|e_\beta|_h^2 = |g_{\alpha\beta}|^2 |e_\alpha|_h^2$ on the overlap. This is in turn equivalent to

$$\phi_\alpha - \phi_\beta = \log |g_{\alpha\beta}|^2 \tag{3.23}$$

which follows immediately from the cocyle relation $g_{\gamma\alpha} g_{\alpha\beta} = g_{\gamma\beta}$.

Definition 3.6 (Chern curvature form)**.** Let h be a hermitian metric on a line bundle $L \to X$. The Chern curvature form $\Theta_h(L)$ is the real, closed $(1,1)$-form defined on U_α by the formula $\Theta_h(L) := \frac{i}{2\pi}\partial\bar\partial\phi_\alpha$. It only depends on h and not on the trivializations of L.

Since $\Theta_h(L)$ is locally exact, it is closed. In particular, it defines an element $[\Theta_h(L)] \in H^2(X, \mathbb{R})$. Moreover, if h, h' are two hermitian metrics on a line bundle $L \to X$, then it follows directly from the definition that there exists a smooth *function* f on X such that $h' = e^f h$. In particular, one has $\Theta_h(L) = \Theta_{h'}(L) + \frac{i}{2\pi}\partial\bar\partial f$, hence $[\Theta_h(L)] = [\Theta_{h'}(L)] \in H^{1,1}(X)$ and we have a well-defined cohomology class $c_1(L) \in H^2(X, \mathbb{R})$.

Lemma 3.7. *The "metric" and "cocycle" definitions of $c_1(L)$ coincide under the identification* (2.15) *between $H^2(X, \mathbb{R})$ and $\check{H}^2(X, \underline{\mathbb{R}})$.*

Proof. Let $c_1^{\mathrm{m}}(L) \in H^2(X, \mathbb{R})$ (resp. $c_1^{\mathrm{c}}(L) \in \check{H}^2(X, \underline{\mathbb{R}})$) be the "metric" (resp. "cocycle") first Chern class of L. We borrow the notation Ψ and Φ from the proof of the isomorphism (2.15).

Recall that $c_1^{\mathrm{m}}(L)$ can be represented by the Chern curvature $\Theta_h(L)$ of any hermitian metric h, e.g. the one defined by (3.22). In that case, we have on U_α the identity $\Theta_h(L) = \frac{i}{2\pi} d\bar{\partial}\phi_\alpha$. Now remember that

$$\bar{\partial}(\phi_\alpha - \phi_\beta) = \bar{\partial} \log |g_{\alpha\beta}|^2 = \bar{\partial}(-4\pi \mathrm{Im}(f_{\alpha\beta})) = -2\pi i d(\overline{f_{\alpha\beta}})$$

where we have used (3.20). Therefore, we have

$$\frac{i}{2\pi}\bar{\partial}(\phi_\alpha - \phi_\beta) = d(\overline{f_{\alpha\beta}}).$$

With the notation of (the proof of) (2.15), the class $\Psi(c_1^{\mathrm{m}}(L))$ is therefore represented by (the conjugate of) $(f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma})_{\alpha\beta\gamma}$, i.e. $\Psi(c_1^{\mathrm{m}}(L)) = c_1^{\mathrm{c}}(L)$. □

Lemma 3.8. *Let $L \to X$ be a line bundle on a compact Kähler manifold X and let $\alpha \in c_1(L)$ be a closed $(1,1)$-form. Then, there exists an hermitian metric h on L such that $\Theta_h(L) = \alpha$.*

Proof. Let h_0 be some background hermitian metric on L. The cohomology class of the closed $(1,1)$-form (or current) $\alpha - \Theta_{h_0}(L)$ is zero, hence by the $\partial\bar{\partial}$-lemma, there exists a function $f \in \mathcal{C}^\infty(X)$ (resp. $f \in L^1(X)$) such that $\alpha - \Theta_{h_0}(L) = \frac{i}{2\pi}\partial\bar{\partial} f$. Set $h := e^{-f} h_0$; it satisfies the requirements. □

3.3 Divisors and line bundles

In this section X is a complex manifold of dimension n.

3.3.1 Divisors

Definition 3.9 (Subvariety)**.** An (analytic) subvariety of X is a closed subset $Y \subset X$ such that for any $x \in X$, there exists an open neighborhood $x \in U \subset X$ such that $Y \cap U$ is the zero set of finitely many holomorphic functions $f_1, \dots, f_k \in \mathcal{O}_X(U)$.

A point $x \in Y$ is said smooth or regular if one can choose the functions f_i such that the Jacobian of the holomorphic map $f = (f_1, \dots, f_k)$ has rank k under a local trivialization. Otherwise, $x \in Y$ is singular.

Definition 3.10 (Dimension)**.** One can show that the set Y_{reg} of regular points of an analytic subvariety $Y \subset X$ is a non-empty complex submanifold of X. We define the dimension of Y by $\dim Y := \dim Y_{\mathrm{reg}}$.

Definition 3.11 (Irreducible subvariety)**.** A subvariety $Y \subset X$ is said irreducible if it cannot be written as the union $Y = Y_1 \cup Y_2$ of two proper analytic subvarieties (i.e. $Y_1 \not\subset Y_2$ and $Y_2 \not\subset Y_1$)

Definition 3.12 (Hypersurface)**.** An hypersurface of X is a subvariety $H \subset X$ of codimension one, i.e. $\dim H = n - 1$.

Definition 3.13 (Divisors and $\mathbb{Q}$-divisors)**.** A divisor (resp $\mathbb{Q}$-divisor) is a formal linear combination $D = \sum_{i=1}^k a_i D_i$ where D_i is an irreducible hypersurface and $a_i \in \mathbb{Z}$ (resp. $a_i \in \mathbb{Q}$). We say that D is *effective* if each a_i is non-negative.

We will admit that similarly to the case of Riemann surfaces, given a meromorphic function $f \in \mathcal{M}_X(X)$, one can attach its divisor of zeros and poles $\mathrm{div}(f) = \sum a_i D_i$ where f vanishes at order a_i along D_i if $a_i \geqslant 0$ and f has a pole of order $-a_i$ if $a_i \leqslant 0$.

Given the local nature of that construction, one can equally define a divisor $\mathrm{div}(s)$ for any meromorphic section of a line bundle L on X.

Definition 3.14 (Equivalence of divisors)**.** We say that two divisors D, D' are linearly equivalent if there exists a meromorphic function $f \in \mathcal{M}_X(X)$ such that $D - D' = \mathrm{div}(f)$. The group of isomorphism classes of divisors under linear equivalence is denoted by $\mathrm{Div}(X)$.

3.3.2 Line bundle associated to a divisor

A smooth hypersurface $H \subset X$ is given locally on a collection of charts $U_\alpha \subset X$ as the zero locus $H \cap U_\alpha = (f_\alpha = 0)$ for some holomorphic function $f_\alpha \in \mathcal{O}_X(U_\alpha)$ such that df_α never vanishes on U_α. It is easy to check that on the overlap $U_{\alpha\beta}$, we have $f_\alpha = g_{\alpha\beta} f_\beta$ for some non-vanishing holomorphic function $g_{\alpha\beta} \in \mathcal{O}_X(U_{\alpha\beta})^*$. Clearly, one has $g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$ on any triple overlap $U_{\alpha\beta\gamma}$. This allows one to define a cocycle $(g_{\alpha\beta}) \in Z^1(X, \mathcal{O}_X^*)$ and it is not hard to check that its class $[(g_{\alpha\beta})] \in H^1(X, \mathcal{O}_X^*)$ is independent of the choice of the functions f_α.

We will admit that the same construction can be carried over similarly if H is merely an analytic hypersurface.

Definition 3.15 (Line bundle associated to an hypersurface)**.** If $H \subset X$ is an analytic hypersurface, the cocycle $[(g_{\alpha\beta})] \in H^1(X, \mathcal{O}_X^*)$ constructed above yields a line bundle that we denote $\mathcal{O}_X(H)$.

Lemma 3.16. *Let $H \subset X$ be an hypersurface. The line bundle $\mathcal{O}_X(H)$ admits a holomorphic section s_H whose divisor is exactly H. It is unique up to an element of $\mathcal{O}_X(X)^*$.*

Proof. This is almost tautological, as one can define s_H by the data of the holomorphic function $\sigma_\alpha = f_\alpha$ on U_α, which satisfies $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$ automatically. As for uniqueness, if s, s' are two such sections, then $\frac{s}{s'}$ is a meromorphic function whose divisor of poles and zero is empty; i.e. it is an element of $\mathcal{O}_X(X)^*$. □

Definition 3.17 (Line bundle associated to a divisor)**.** Let $D = \sum_{i=1}^k a_i D_i$ be a divisor. We define $\mathcal{O}_X(D) := \mathcal{O}_X(D_1)^{\otimes a_1} \otimes \cdots \otimes \mathcal{O}_X(D_k)^{\otimes a_k}$.

In terms of cocycles, if $(f_\alpha^{(i)} = 0)$ is the equation of $D_i \cap U_\alpha$, then the cocycle associated to $\mathcal{O}_X(D)$ is simply $\prod_{i=1}^k \left(\frac{f_\alpha^{(i)}}{f_\beta^{(i)}}\right)^{a_i}$. Moreover, $\mathcal{O}_X(D)$ admits a meromorphic section $s_D := \prod_{i=1}^k s_{D_i}^{\otimes a_i}$ which satisfies $\mathrm{div}(s_D) = D$.

Lemma 3.18. *Let L be a line bundle on X endowed with a meromorphic section s. Then L is isomorphic to $\mathcal{O}_X(\mathrm{div}(s))$.*

Proof. The meromorphic section s correspond to meromorphic functions σ_α on U_α such that $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$ if $g_{\alpha\beta}$ are the transition functions of L. Set $D = \mathrm{div}(s)$. Since, locally, $D \cap$

$U_\alpha = \mathrm{div}(\sigma_\alpha)$, the line bundle $\mathcal{O}_X(D)$ can be defined by the cocycle $\frac{\sigma_\alpha}{\sigma_\beta}$ which coincides with $g_{\alpha\beta}$. This proves the lemma. □

Lemma 3.19. *Let D be a divisor. Then $\mathcal{O}_X(D)$ is isomorphic to the trivial line bundle $\mathcal{O}_X$ if and only if $D = \mathrm{div}(f)$ for some meromorphic function $f \in \mathcal{M}_X(X)$.*

Proof. If $D = \mathrm{div}(f)$, then one chooses as local equation $(f = 0)$ on U_α and the cocycle $g_{\alpha\beta}$ is noting but $\frac{f}{f} \equiv 1$. Conversely assume that we have a trivializing section $s \in H^0(X, \mathcal{O}_X(D))$. Then s_D/s is a meromorphic function whose divisor is $\mathrm{div}(s_D) = D$. □

Corollary 3.20. *The map*

$$\begin{aligned} \mathrm{Div}(X) &\longrightarrow \mathrm{Pic}(X) \\ D &\longmapsto \mathcal{O}_X(D) \end{aligned}$$

is an injective morphism of abelian groups. Moreover, the image consists exactly of line bundles admitting a non-zero meromorphic section.

Proof. The fact that the map is well-defined and injective follows from Lemma 3.19. The description of the image follows from Lemmas 3.16-3.18. □

Remark 3.21. If X is a projective manifold, then the above map is surjective, hence isomorphic. Indeed, if L is a line bundle and H is an ample hypersurface, then for $p \gg 1$, $L \otimes H^{\otimes p}$ is globally generated, hence it has a non-zero section s. Then, $s/s_H^{\otimes p}$ is a non-zero rational section of L.

Lemma 3.22. *Let D be a divisor, let $L := \mathcal{O}_X(D)$ be the associated line bundle and let $\mathcal{L}$ be the sheaf of sections of L. Then $\mathcal{L}$ is locally free of rank one and given an open set $U \subset X$, one has a natural correspondence*

$$\mathcal{L}(U) \simeq \{f \in \mathcal{M}_X(U);\ \mathrm{div}(f) \geqslant -D|_U\}.$$

Proof. Let s_D be a meromorphic section of L such that $\mathrm{div}(s_D) = D$. Let $\mathcal{F}$ be the sheaf defined by $\mathcal{F}(U) = \{f \in \mathcal{M}_X(U);\ \mathrm{div}(f) \geqslant -D|_U\}$. We have a sheaf morphism $\mathcal{L} \to \mathcal{F}$ given by $s \mapsto \frac{s}{s_D}$ and its inverse is $t \mapsto ts_D$. It remains to see that $\mathcal{L}$ is locally free or, equivalently, that $\mathcal{F}$ is locally free. Consider the covering of X by open sets U_α where $D|_{U_\alpha} = \mathrm{div}(f_\alpha)$ for f_α a meromorphic function on U_α. Then $\frac{1}{f_\alpha} \in \mathcal{F}(U_\alpha)$ and one has clearly $\mathcal{F}|_{U_\alpha} = \frac{1}{f_\alpha}\mathcal{O}_{U_\alpha}$. □

3.4 Basics on intersection theory

Let X be a compact Kähler manifold of dimension n. If $\alpha_1, \ldots, \alpha_n \in H^2(X, \mathbb{R})$, we set

$$\alpha_1 \cdot \ldots \cdot \alpha_n = \int_X \omega_1 \wedge \ldots \wedge \omega_n$$

where $\omega_i \in \alpha_i$ is an arbitrary representative of its de Rham cohomology class. This defines a multi-linear symmetric product on $H^2(X, \mathbb{R})$ called the intersection product.

Let $L \to X$ be a holomorphic line bundle which admits a non-zero section $s \in H^0(X, L)$. We pick a hermitian metric h on L and set $D := \mathrm{div}(s)$. We have the following result

Theorem 3.23. *Assume that D is smooth. For any $(n-1, n-1)$-form ψ on X, we have*

$$\frac{i}{2\pi}\int_X (\log |s|_h^2) \cdot \partial\bar{\partial}\psi = \int_D \psi|_D - \int_X \Theta(L,h) \wedge \psi.$$

Remark 3.24. The result holds more generally without the assumption that D is smooth and is due to Lelong. The formula then reads

$$\frac{i}{2\pi}\int_X (\log |s|_h^2) \cdot \partial\bar{\partial}\psi = \int_{D_{\mathrm{reg}}} \psi|_{D_{\mathrm{reg}}} - \int_X \Theta(L,h) \wedge \psi$$

if D is reduced, and generalized to the non-reduced case in the obvious way. The additional ingredient needed is that D can be locally expressed as a (finite) ramified cover of $\mathbb{C}^{n-1}$.

Corollary 3.25. *Let $\alpha_1, \ldots, \alpha_{n-1} \in H^2(X, \mathbb{R})$ and let D a smooth hypersurface. Then*

$$\alpha_1 \cdot \ldots \cdot \alpha_{n-1} \cdot c_1(D) = \alpha_1|_D \cdot \ldots \cdot \alpha_{n-1}|_D.$$

Proof of Corollary 3.25. Choose representatives ω_i of α_i, set $\psi := \omega_1 \wedge \ldots \wedge \omega_{n-1}$ which is $\partial\bar{\partial}$-closed. Since $\Theta(L,h) \in c_1(D)$, the result follows from Theorem 3.23. □

Corollary 3.26. *Let L be a line bundle such that $c_1(L) = 0 \in H^2(X, \mathbb{R})$. Then $H^0(X,L) = 0$ unless L is trivial.*

Proof of Corollary 3.26. Argue by contradiction and let $s \in H^0(X,L)$ be a non-zero section. Write $D = \sum a_i D_i$ with $a_i > 0$, D_i irreducible. Let $\alpha \in H^2(X,\mathbb{R})$ be a Kähler class and $\omega \in \alpha$ be a Kähler representative. By Theorem 3.23 and Remark 3.24, we have

$$c_1(D) \cdot \alpha^{n-1} = \sum_i a_i \int_{D_{i,\mathrm{reg}}} \omega|_{D_{i,\mathrm{reg}}}^{n-1} > 0,$$

a contradiction with $c_1(D) = 0$. □

Remark 3.27 (Alternative proof of Corollary 3.26)**.** By Theorem 5.2, there exists a hermitian metric h on L such that $\Theta(L,h) = 0$. But then one has for any holomorphic section s of L

$$\partial\bar{\partial}|s|^2 = \langle D's, D's\rangle + \langle s, 2\pi i \Theta(L,h)s\rangle = \langle D's, D's\rangle$$

where D' is the $(1,0)$-part of the Chern connection of (L,h). In particular, we get for any Kähler metric ω

$$0 = \int_X \Delta|s|^2 \cdot \omega^n = n \int_X |D's|_{h,\omega}^2 \omega^n$$

hence $D's = 0$. In particular, $|s|^2$ is constant and either s vanishes identically or it never vanishes.

Proof of Theorem 3.23. Let us first observe that $\log |s|_h^2$ is integrable with respect to any volume form on X, so that our integral is well-defined.

Next, choose a partition of unity χ_α subordinate to a covering (U_α) where $U_\alpha \simeq \Delta^n$ is a polydisk and $D|_{U_\alpha} = (z_1^\alpha = 0)$ under the previous identification. Writing $\psi = \sum_\alpha \chi_\alpha \psi$, one sees that it is enough to prove the formula when ψ has compact support in some U_α. Therefore, one will fix such an index α and set $\psi := \chi_\alpha \psi$, $U := U_\alpha$, $z_i := z_i^\alpha$.

One can choose a trivialization e of $L|_U$ such that $s|_U = z_1 \cdot e$ and set $\phi := -\log |e|_h^2$ so that $\Theta(L,h)|_U = \frac{i}{2\pi}\partial\bar\partial\phi$. Then,

$$\begin{aligned}
\frac{i}{2\pi}\int_U (\log |s|_h^2)\cdot \partial\bar\partial\psi &= \frac{i}{2\pi}\int_U (\log|z_1|^2 - \phi)\cdot\partial\bar\partial\psi \\
&= \lim_{\varepsilon\to 0}\frac{i}{2\pi}\int_U (\log(|z_1|^2+\varepsilon^2)-\phi)\cdot\partial\bar\partial\psi \\
&= \lim_{\varepsilon\to 0}\frac{i}{2\pi}\int_U \partial\bar\partial\log(|z_1|^2+\varepsilon^2)\wedge\psi - \int_U \Theta(L,h)\wedge\psi,
\end{aligned}$$

thank to Stokes formula. Therefore, all we have to prove is

$$\lim_{\varepsilon\to 0}\frac{i}{2\pi}\int_U \partial\bar\partial\log(|z_1|^2+\varepsilon^2)\wedge\psi = \int_D \psi|_D. \tag{3.24}$$

Now we have

$$\partial\bar\partial\log(|z_1|^2+\varepsilon^2) = \partial\left(\frac{z_1 d\bar z_1}{|z_1|^2+\varepsilon^2}\right) = \frac{\varepsilon^2}{(|z_1|^2+\varepsilon^2)^2}\cdot dz_1\wedge d\bar z_1.$$

Now if f is a smooth function on the disk (of radius two) in $\mathbb{C}$, then

$$\begin{aligned}
I_{f,\varepsilon} := \frac{1}{2\pi}\int_{|z|<1}\frac{\varepsilon^2}{(|z|^2+\varepsilon^2)^2}\cdot f(z) i dz\wedge d\bar z &= \frac{\varepsilon^2}{2\pi}\int_0^{2\pi} d\theta\int_0^1 f(re^{i\theta})\frac{2rdr}{(r^2+\varepsilon^2)^2} \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{\varepsilon^{-1}} f(\varepsilon s e^{i\theta})\frac{2sds}{(s^2+1)^2}
\end{aligned}$$

where one has used the change of variable $r = \varepsilon s$. Since $\frac{s}{(s^2+1)^2} = O(\frac{1}{1+s^3})$, Lebesgue dominated convergence theorem shows

$$\lim_{\varepsilon\to 0} I_{f,\varepsilon} = f(0)\int_0^{+\infty}\frac{2sds}{(s^2+1)^2} = f(0)\int_1^{+\infty}\frac{du}{u^2} = f(0).$$

Therefore, if one write $z = (z_1, \underline{z})$ and

$$\psi = a(z) dz_2\wedge d\bar z_2\wedge\ldots\wedge dz_n\wedge d\bar z_n + \tau,$$

where τ involves only summands where either dz_1 or $d\bar z_1$ appears, then

$$\partial\bar\partial\log(|z_1|^2+\varepsilon^2)\wedge\psi = \frac{\varepsilon^2 a(z)}{(|z_1|^2+\varepsilon^2)^2}\cdot dz_1\wedge d\bar z_1\wedge\ldots\wedge dz_n\wedge d\bar z_n$$

hence

$$\lim_{\varepsilon\to 0}\frac{i}{2\pi}\int_U \partial\bar\partial\log(|z_1|^2+\varepsilon^2)\wedge\psi = \int_{\underline{z}\in\Delta^{n-1}} a(0,\underline{z}) dz_2\wedge d\bar z_2\wedge\ldots\wedge dz_n\wedge d\bar z_n = \int_D \psi|_D,$$

hence (3.24) holds. This proves the theorem. □

Lemma 3.28. *Let $f : Y \to X$ be a map between compact complex manifolds of dimension n and let $\alpha_1, \ldots, \alpha_n \in H^2(X, \mathbb{C})$.*

1. *If f is finite of degree d, then*
$$(f^*\alpha_1 \cdot \ldots \cdot f^*\alpha_n) = d(\alpha_1 \cdot \ldots \cdot \alpha_n).$$

2. *If f is bimeromorphic, i.e. if there exists a Zariski open set $U \subset X$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is an isomorphism, then*
$$(f^*\alpha_1 \cdot \ldots \cdot f^*\alpha_n) = (\alpha_1 \cdot \ldots \cdot \alpha_n).$$

Proof. Pick representatives $\omega_i \in \alpha_i$. Outside of the ramification divisor $R \subset Y$ (which has Lebesgue measure zero), f is an étale covering hence

$$\begin{aligned} \int_Y f^*\omega_1 \wedge \ldots \wedge f^*\omega_n &= \int_{Y \setminus Z} f^*\omega_1 \wedge \ldots \wedge f^*\omega_n \\ &= d \int_{X \setminus f(Z)} \omega_1 \wedge \ldots \wedge \omega_n \\ &= d \int_X \omega_1 \wedge \ldots \wedge \omega_n \end{aligned}$$

by Lemma 3.29. The second item is proved similarly (with $d = 1$ for the étale covering $f|_{f^{-1}(U)}$). □

Lemma 3.29. *Let $g : M \to N$ be a smooth topological cover of degree d between compact oriented differentiable manifolds of real dimension k. For any k-form ω on N, we have*

$$\int_M g^*\omega = d \int_N \omega.$$

Proof. Fix a covering $N = \cup N_\alpha$ such that $g^{-1}(N_\alpha) = \sqcup_\beta M_{\alpha\beta}$ is a disjoint union of d connected open sets $(M_{\alpha\beta})_{\beta=1\ldots d}$ such that $g|_{M_{\alpha\beta}} : M_{\alpha\beta} \to N_\alpha$ is an isomorphism. Fix a partition of unity (χ_α) subordinate to (N_α). By the chance of variable formula, we have $\int_{N_\alpha} \chi_\alpha \omega = \int_{M_{\alpha\beta}} g^*(\chi_\alpha \omega)$, hence we have

$$\begin{aligned} \int_N \omega &= \sum_\alpha \int_{N_\alpha} \chi_\alpha \omega \\ &= \sum_\alpha \frac{1}{d} \sum_\beta \int_{M_{\alpha\beta}} g^*(\chi_\alpha \omega) \\ &= \frac{1}{d} \sum_\alpha \int_{g^{-1}(N_\alpha)} g^*(\chi_\alpha) g^*\omega \\ &= \frac{1}{d} \sum_\alpha \int_M g^*(\chi_\alpha) g^*\omega \\ &= \frac{1}{d} \int_M g^*\omega. \end{aligned}$$

□

4 The projective space

4.1 The line bundles $\mathcal{O}_{\mathbb{P}^n}(k)$

Let $\mathbb{P}^n$ be the space of lines through the origin in $\mathbb{C}^{n+1}$ and let $L := \mathcal{O}_{\mathbb{P}^n}(1)$ be the dual of the tautogical bundle $\mathcal{O}_{\mathbb{P}^n}(-1) = \{(x,v) \in \mathbb{P}^n \times \mathbb{C}^{n+1}; v \in x\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$. In other words, if $x = [z_0 : \ldots : z_n] \in \mathbb{P}^n$, then the fiber L_x consists of all linear forms on the line $\mathbb{C}(z_0, \ldots, z_n)$.

One can cover X by the charts $U_i = \{[z_0 : \ldots : z_n] \in \mathbb{P}^n; z_i \neq 0\}$ which are isomorphic to $\mathbb{C}^n$ via the coordinates $w_0 = \frac{z_0}{z_i}, \ldots, \widehat{w}_i, \ldots, w_n = \frac{z_n}{z_i}$. Each linear projection

$$(\lambda z_0, \ldots, \lambda z_i) \mapsto \lambda z_i$$

defines a section e_i of L which is non-zero exactly on U_i, and one has $e_j = \frac{z_j}{z_i} \cdot e_i$ on U_{ij}. This shows that the transitions function of L on $U_i \cap U_j$ are $g_{ij} = \frac{z_j}{z_i}$.

If we set $\phi_i := -\log \frac{|z_i|^2}{\|z\|^2}$ on U_i, then $\phi_i - \phi_j = \log |g_{ij}|^2$ on U_{ij} hence $e^{-\phi_i}$ defines an hermitian metric h_{FS} on L. Its curvature can be computed on U_i using the w-coordinate as

$$\Theta_{h_{\mathrm{FS}}}(L)|_{U_i} = \frac{i}{2\pi} \partial\bar{\partial} \log(1 + \|w\|^2).$$

One has

$$\Theta_{h_{\mathrm{FS}}}(L)|_{U_i} = \frac{1}{2\pi} \frac{1}{(1+\|w\|^2)^2} \sum_{j,k} a_{jk} i dz_j \wedge d\bar{z}_k, \quad \text{with} \quad a_{jk} = (1+\|w\|^2)\delta_{jk} - w_k \bar{w}_j.$$

The hermitian matrix (a_{jk}) has eigenvalues $(1+\|w\|^2)$ with multiplicity $(n-1)$ and 1 with multiplicity 1, hence it is definite positive. Hence

$$\omega_{\mathrm{FS}} := \Theta_{h_{\mathrm{FS}}}(L)$$

is a Kähler form, called Fubini-Study metric, and $\mathcal{O}_{\mathbb{P}^n}(1)$ is positive. One can check that it satisfies

$$\int_{\mathbb{P}^n} \omega_{\mathrm{FS}}^n = 1. \tag{4.25}$$

Alternatively, one can describe h_{FS} more intrinsically as the dual metric of the restriction to $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ of the euclidean metric on $\mathbb{C}^{n+1}$. That is, if $x \in \mathbb{P}^n$ and $\phi \in \mathcal{O}_{\mathbb{P}^n}(1)_x = x^\vee$,

$$\|\phi\|^2_{h_{\mathrm{FS}}} = \sup\{|\phi(v)|^2 \mid v \in x, \|v\|^2 = 1\}.$$

If $x = [z_0 : \ldots : z_n] \in U_i$, we recover

$$\|e_i(x)\|^2_{h_{\mathrm{FS}}} = \sup\{|\lambda z_i|^2 | \sum_k |\lambda z_k|^2 = 1\} = \frac{|z_i|^2}{\sum_k |z_k|^2}.$$

Since $H^2(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$ is generated by $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ (cf Proposition 4.3 below), we have that every class $\alpha \in H^{1,1}(X, \mathbb{R})$ is either zero, Kähler, or anti-Kähler.

Remark 4.1. Let A be a definite positive hermitian matrix of size $n+1$, and let h_A be the dual of the hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ induced by restriction (e.g. $h_{I_{n+1}} = h_{\mathrm{FS}}$). The square root of A induces an isomorphism $f \in \mathrm{Aut}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1))$ such that $f^* h_{\mathrm{FS}} = h_A$. In particular, $\Theta_{h_A}(\mathcal{O}_{\mathbb{P}^n}(1)) = f^*\omega_{\mathrm{FS}}$ is definite positive.

Proposition 4.2. *For any $k \geqslant 1$, there is an isomorphism*

$$H^0(\mathbb{P}^n, \mathcal{O}(k)) \simeq \mathrm{Sym}^k(\mathbb{C}^{n+1})^*$$

with the space of homogenous polynomials of degree k in $n+1$ variables. It has dimension $\binom{n+k}{n}$.

Proof. We have seen above that each degree one monomial z_i on $\mathbb{C}^{n+1}$ for $i = 0, \ldots, n$ induces a global section e_i of $\mathcal{O}(1)$. Therefore, a degree k monomial $z_0^{a_0} \cdots z_n^{a_n}$ induces the global section $e_0^{\otimes a_0} \otimes \ldots \otimes e_n^{\otimes a_n}$ of $\mathcal{O}(\sum a_i) = \mathcal{O}(k)$. This gives a map

$$\mathrm{Sym}^k(\mathbb{C}^{n+1})^* \to H^0(\mathbb{P}^n, \mathcal{O}(k)), \quad z_0^{a_0} \cdots z_n^{a_n} \mapsto e_0^{\otimes a_0} \otimes \ldots \otimes e_n^{\otimes a_n}$$

which is easily seen to be injective e.g. by observing that $e_0^{\otimes a_0} \otimes \ldots \otimes e_n^{\otimes a_n}$ vanishes at order exactly a_i along the hyperplane of $\mathbb{P}^n$ defined by the vanishing of the i-th homogeneous coordinate.

If one prefers to think in term of local trivializations, recall that $\mathcal{O}(k)$ is trivialized by $e_i^{\otimes k}$ on U_i. Now, a homogeneous polynomial P of degree k yields holomorphic functions $f_i = \frac{P}{z_i^k}$ on U_i. These functions satisfy $f_i = \frac{z_j^k}{z_i^k} f_j$ on the overlap $U_i \cap U_j$ and the global section of $\mathcal{O}(k)$ is given by the local sections $f_i e_i^{\otimes k}$ on U_i who do indeed glue.

Next, let $s \in H^0(\mathbb{P}^n, \mathcal{O}(k))$. We view s as a collection of holomorphic functions f_i on U_i such that $f_i = \frac{z_j^k}{z_i^k} f_j$ on $U_i \cap U_j$. Let $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ the usual projection map. Then the functions $z_i^k f_i \circ p$ glue to a global holomorphic function $\hat{f}$ on $\mathbb{C}^{n+1} \setminus \{0\}$ satisfying $\hat{f}(\lambda z) = \lambda^k \hat{f}(z)$. By Hartogs extension theorem, $\hat{f}$ extends to a holomorphic function on $\mathbb{C}^{n+1}$ which is homogeneous of degree k. Looking at the power series expansion of $\hat{f}$ near the origin, the homogeneity forces $\hat{f}$ to be a homogeneous polynomial of degree k. Since $f_i = \frac{\hat{f}}{z_i^k}$, this shows the claim.

To finish the proof of the proposition, one needs to check that the dimension of the space of homogenous polynomials of degree k in $n+1$ variables is $\binom{n+k}{n}$. We have to counts the number of monomials $z_0^{a_0} \cdots z_n^{a_n}$ with $\sum_{i=0}^n a_i = k$. If one represents $z_0^{a_0} \cdots z_n^{a_n}$ as

$$z_0 \cdots z_0 * z_1 \cdots z_1 * \cdots * z_n \cdots z_n,$$

we see that the choice of each monomial can be represented uniquely with $n+k$ symbols involving exactly n symbols $*$ and k symbols among the $z_0, \ldots, z_n$. Now each monomial is determined by the choice the n symbols $*$ among the $n+k$, hence the result. $\square$

Proposition 4.3. *The Picard group of $\mathbb{P}^n$ is isomorphic to $\mathbb{Z}$, generated by $\mathcal{O}_{\mathbb{P}^n}(1)$.*

Proof. Recall that $H^1(\mathbb{P}^n, \mathbb{Z}) = 0$ and $H^2(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$. In particular, we have vanishing $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0$ thanks to the Hodge decomposition. Now, the long exact sequence associated to the exponential exact sequence yields a group isomorphism

$$\psi : H^1(\mathbb{P}^n, \mathcal{O}^*_{\mathbb{P}^n}) \longrightarrow H^2(X, \mathbb{Z}),$$

cf (3.21). It remains to see that $\mathcal{O}(1)$ is primitive. This follows from the fact that the first Chern class $c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \in H^2(X, \mathbb{R})$ is represented by the Fubini-Study metric ω_{FS} which satisfies (4.25). □

4.2 The Euler exact sequence

The goal of this paragraph is to investigate the (co)tangent bundle of $\mathbb{P}^n$. The key result is the so-called Euler exact sequence which givens an explicit presentation of $\Omega^1_{\mathbb{P}^n}$.

Proposition 4.4 (Euler exact sequence)**.** *We have an exact sequence*

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \xrightarrow{f} \mathcal{O}_{\mathbb{P}^n}(-1) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) \xrightarrow{g} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

Proof. The map g comes from the natural pairing between a bundle and its dual. We will define the map f in local charts, check that the sequence is exact there, and then check that the maps can be glued.

Let us fix some notation first. We considered earlier the basis of sections $e_0, \ldots, e_n$ of $H^0(\mathbb{P}^n, \mathcal{O}(1))$. Next, on each U_i, we have the coordinates $w_k = \frac{z_k}{z_i}$ for $k = 0, \ldots, \hat{i}, \ldots, n$. It is convenient to set $w_i = 1$. Finally, $\mathcal{O}(-1)|_{U_i}$ admits the trivialization $e_i^* = (w_0, \ldots, w_n) \in \mathbb{C}^{n+1}$ which satisfies $e_k(e_i^*) = w_k$ for each index k.

The one-forms $(dw_k)_{k \neq i}$ give rise to a holomorphic frame for $\Omega^1_{\mathbb{P}^n}|_{U_i}$ and we define the map

$$\begin{array}{cccc} f_i: & \Omega^1_{\mathbb{P}^n}|_{U_i} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(-1)|_{U_i} \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) \\ & dw_k & \mapsto & e_i^* \otimes (w_i e_k - w_k e_i) \end{array}$$

whose image has rank n and lies in the kernel of g. This implies that $\mathrm{Ker}(g|_{U_i}) = \mathrm{Im}(f_i)$.

Now we are left to checking that the morphisms f_i glue to a morphism f. For this, we consider another standard open set U_j with coordinates $u_k = \frac{z_k}{z_j}$, yielding a map f_j. We have $\frac{e_i^*}{e_j^*} = \frac{z_j}{z_i} = w_j$. We need to prove that

$$f_i(du_k) = f_j(du_k) := e_j^* \otimes (u_j e_k - u_k e_j).$$

For this, we write $u_k = \frac{w_k}{w_j}$ so that $du_k = \frac{1}{w_j} dw_k - \frac{w_k}{w_j^2} dw_j$. Therefore, we find

$$f_i(du_k) = w_j e_j^* \otimes \frac{1}{w_j} \Big((w_i e_k - w_k e_i) - \frac{w_k}{w_j} (w_i e_j - w_j e_i) \Big) = e_j^* \otimes (u_j e_k - u_k e_j)$$

since $w_i = u_j = 1$ and $\frac{w_k}{w_j} = u_k$. This completes the proof of the proposition. □

Remark 4.5. By dualizing the Euler exact sequence, we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1))^* \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0, \tag{4.26}$$

which can be interpreted as follows. The homogeneous vector fields $\xi_{ij} := z_j \frac{\partial}{\partial z_i}$ on $\mathbb{C}^{n+1}$ descend to $\mathbb{P}^n$ and generate $T_{\mathbb{P}^n}$. For $i \neq j$, ξ_{ij} integrates to an action of $\mathbb{C}$ on $\mathbb{C}^{n+1}$ given by $t \cdot (z_1, \ldots, z_n) = (z_1, \ldots, z_{i-1}, z_i + tz_j, z_{i+1}, \ldots, z_n)$ if, say, $i < j$. If $i = j$, then ξ_{ii} integrates to an action of $\mathbb{C}$ on $\mathbb{C}^{n+1}$ given by $t \cdot (z_1, \ldots, z_n) = (z_1, \ldots, z_{i-1}, e^t \cdot z_i, z_{i+1}, \ldots, z_n)$. Both actions commute with the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ and descend to $\mathbb{P}^n$. Finally, the Euler vector field $\sum_{i=0}^n z_i \frac{\partial}{\partial z_i} = \sum_i \xi_{ii}$ descends to the trivial one as it corresponds to the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$.

4.3 Maps to projective space

Let X be a compact complex manifold and let $L \to X$ be a line bundle on X. Assume that $V := H^0(X, L) \neq 0$. Set $N + 1 := \dim V$ and define

$$\mathbb{B}(L) := \bigcap_{s \in V} s^{-1}(0) = \{x \in X; s(x) = 0 \, \forall s \in V\}$$

which is called the base locus of L. By assumption, $Z \subsetneq X$ is a proper analytic subset of X.

We fix a basis $s_0, \ldots, s_N$ of V. For $x \in X$, pick a local trivialization e_L of L near x and write $s_i(x) = f_i(x)e_L$. This defines holomorphic functions f_i near x which depend on the choice of e_L. However, if $x \notin \mathbb{B}(L)$, one can consider the point $[f_0(x) : \ldots : f_N(x)] \in \mathbb{P}^N$ which is well-defined and does not depend on the choice of e_L. This shows that there exists a well-defined holomorphic map

$$\Phi_L : X \setminus \mathbb{B}(L) \longrightarrow \mathbb{P}^N$$

which we will (abusively) write as $\Phi_L(x) = [s_0(x) : \ldots : s_N(x)]$. This maps depends on the choice of a basis of V, and any other choice of basis yields a map Φ'_L which is related to Φ_L by the relation $\Phi'_L = g \circ \Phi_L$ for some $g \in \mathrm{PGL}(N+1, \mathbb{C})$.

One can also define Φ_L in a more intrinsic way as follows. Consider for $x \in X$ the evaluation map $\mathrm{ev}_x : H^0(X, L) \to L_x$. Whenever $x \notin \mathbb{B}(L)$, the kernel $H_x := \mathrm{Ker}(\mathrm{ev}_x)$ defines a hyperplane of V. The set $\{f \in V^*; f|_{H_x} \equiv 0\}$ defines a line $\ell_x \subset V^*$ and one can define $\Phi_L(x) = \ell_x \in \mathbb{P}(V^*)$.

Lemma 4.6. *The map Φ_L induces an isomorphism*

$$\Phi_L^* \mathcal{O}_{\mathbb{P}^N}(1) \simeq L|_{X \setminus \mathbb{B}(L)}.$$

Proof. One can assume that $h^0(X, L) \geqslant 2$, otherwise there is nothing to prove (both bundles in the sought isomorphism are trivial). Consider the divisor $D = \mathrm{div}(s_0)$; it is not included in $\mathbb{B}(L)$ (in particular, it is non-empty). The hyperplane $H = (z_0 = 0) \in \mathbb{P}^N$ is the divisor of zeros of the section $e_0 \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ (cf § 4.2). Clearly, $\Phi_L^* e_0$ is a section of the line bundle $\Phi_L^* \mathcal{O}_{\mathbb{P}^N}(1)$ on $X \setminus \mathbb{B}(L)$ and it is enough to show that

$\operatorname{div}(\Phi_L^* e_0) = D|_{X \setminus \mathbb{B}(L)}$ by Lemma 3.18. Now this is a local property. As sets, we have $(\Phi_L^* e_0 = 0) = D|_{X\setminus\mathbb{B}(L)}$ but we need to prove the equality as divisors. Pick $x \in D \setminus \mathbb{B}(L)$; there exists $i \geqslant 1$ such that $s_i(x) \neq 0$ so that a small neighborhood W of x is sent to $U_i = (z_i \neq 0) \subset \mathbb{P}^N$ by Φ_L. Write $s_j = f_j s_i$ on W so that $\Phi_L = (f_0, \dots, \hat{1}, \dots, f_N)$ with the usual coordinates on U_i. Since $e_0 = w_0 e_i$, we find on W

$$\Phi_L^* e_0 = (w_0 \circ \Phi_L) \cdot \Phi_L^* e_i = f_0 \cdot \Phi_L^* e_i$$

so that $\operatorname{div}(\Phi_L^* e_i)|_W = \operatorname{div}(f_0) = D|_W$. The lemma follows. □

Lemma 4.7. *Let L be a line bundle on a compact complex manifold X. Assume that $V := H^0(X, L)$ is non-empty.*

1. *Φ_L is defined everywhere if for any $x \in X$, there exists $s \in V$ such that $s(x) \neq 0$.*

2. *Φ_L is injective iff given any two distinct points $x, y \in X \setminus \mathbb{B}(L)$, there exists $s \in V$ such that $s(x) \neq 0$ and $s(y) = 0$.*

3. *Φ_L is a local biholomorphism onto its image iff given any $x \in X \setminus \mathbb{B}(L)$ and $v \in T_{X,x} \setminus 0$, there exists $s \in V$ such that $s(x) = 0$ and $ds(v) \neq 0$.*

The third point deserves some explanation. If $s \in V$ vanishes at x, then given a local trivialization e_L of L near x one can write $s = f e_L$ with f a holomorphic function vanishing near x. We define that $ds(v) := df(v) \otimes e_L$ as an element of $\Omega^1_{X,x} \otimes L_x$. It is well-defined independently of the choice of e_L. Indeed, if $e'_L = h^{-1} e_L$ is another trivialization, then $s = f' e'_L$ with $f' = hf$. We get $df'(v) = h(x) df(v)$ hence $df'(v) \otimes e'_L = df(v) \otimes e_L$.

Proof. The first point is clear.

As for the second point, choose sections $s_1, s_2 \in V$ such that $s_1(x) = 0, s_2(y) = 0$ and $s_2(x) \neq 0, s_1(y) \neq 0$ and complete to a basis (s_i) of V. Then $\Phi_L(x) = [0 : 1 : *]$ while $\Phi_L(y) = [1 : 0 : *]$ thus defining two distinct points in projective space.

Let us get to the third point. For $x \notin \mathbb{B}(L)$, pick a basis $(s_1, \dots, s_N)$ of the hyperplane $V_x := \{s \in V, s(x) = 0\}$ that we complete to a basis of V by adding one element $s_{N+1} \in V \setminus V_x$. Near x, one can write $s_i = f_i s_{N+1}$ for $i = 1, \dots, N$ where f_i is a holomorphic function vanishing at x. The map Φ_L near x can be read as the map to $\mathbb{C}^N$ given by $(f_1, \dots, f_N)$. By assumption, given any non-zero vector $v \in T_{X,x}$, there exists $s \in V_x$ such that $ds_x(v) \neq 0$. Since V_x is generated by the $s_i = f_i s_{N+1}$ for $i = 1, \dots, N$, this means that there exists one such index such that $df_i(v) \neq 0$. In particular, this shows that $d\Phi_L(v) \neq 0$. The lemma now follows from the inverse function theorem. □

5 Some applications of Hodge theory

5.1 Lefschetz theorem on $(1,1)$-classes

Let X be a complex manifold of dimension n and let $0 \leqslant k \leqslant n$ be an integer. Given a k-form α, its $(0,k)$-component $\alpha_{0,k} = \pi^{0,k}\alpha$ satisfies $(d\alpha)_{0,k+1} = \bar{\partial}\alpha_{0,k}$. In particular, if

$d\alpha = 0$ then $\bar\partial \alpha_{0,k} = 0$. Similarly, if $\alpha = d\beta$ is d-exact, then $\alpha_{0,k} = \bar\partial(\beta_{0,k-1})$ is $\bar\partial$-exact. This shows that there exists a natural map

$$H^k_{\mathrm{dR}}(X, \mathbb{C}) \to H^{0,k}(X). \tag{5.27}$$

Moreover, if X is compact and Kähler, then the above map coincides with the projection map arising from the Hodge decomposition theorem. Next, the inclusion of sheaves $\mathbb{C} \to \mathcal{O}_X$ yields a map $H^k(X, \mathbb{C}) \to H^k(X, \mathcal{O}_X)$ in Čech cohomology. These two maps can be related as follows.

Lemma 5.1. *We have a commutative diagram*

$$\begin{tikzcd}
H^k_{\mathrm{dR}}(X, \mathbb{C}) \arrow[r] \arrow[d, "\cong"] & H^{k,0}(X) \arrow[d, "\cong"] \\
H^k(X, \mathbb{C}) \arrow[r] & H^k(X, \mathcal{O}_X)
\end{tikzcd}$$

where the vertical arrows are the de Rham-Weil isomorphisms (2.14) *and* (2.19).

Proof. The following diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d, hook] & \mathcal{A}^0_X \arrow[r, "d"] \arrow[d, "="] & \mathcal{A}^1_X \arrow[r, "d"] \arrow[d, "\pi^{0,1}"] & \mathcal{A}^2_X \arrow[r, "d"] \arrow[d, "\pi^{0,2}"] & \cdots \arrow[r, "d"] & \mathcal{A}^k_X \arrow[d, "\pi^{0,k}"] \\
\mathcal{O}_X \arrow[r] & \mathcal{A}^0_X \arrow[r, "\bar\partial"] & \mathcal{A}^{0,1}_X \arrow[r, "\bar\partial"] & \mathcal{A}^{0,2}_X \arrow[r, "\bar\partial"] & \cdots \arrow[r, "\bar\partial"] & \mathcal{A}^{0,k}_X
\end{tikzcd}$$

is commutative thanks to the above discussion. Using the de Rham Weil isomorphism theorem, we see that the map $H^k(X, \mathbb{C}) \to H^k(X, \mathcal{O}_X)$ corresponds to the map (5.27) described above. $\square$

We have seen in the previous section (cf Lemma 3.7 and (3.21)) that if $L \to X$ is a complex manifold, then $c_1(L) \in H^2(X, \mathbb{R})$ lies in the image of $H^2(X, \underline{\mathbb{Z}}) \to H^2(X, \underline{\mathbb{R}})$. Moreover, it is clear from the metric definition of $c_1(L)$ that it can be represented by a (d-closed) $(1,1)$-form. Actually, the converse holds.

More precisely, we say that a class $\alpha \in H^2(X, \mathbb{R})$ is an *integral class* if under the isomorphism (2.15) between $H^2(X, \mathbb{R})$ and $H^2(X, \underline{\mathbb{R}})$, we have $\alpha \in \mathrm{Im}(j : H^2(X, \underline{\mathbb{Z}}) \to H^2(X, \underline{\mathbb{R}}))$. Then we have the following result.

Theorem 5.2 (Lefschetz theorem on $(1,1)$-classes)**.** *Let X be a complex manifold, and let $\omega \in \mathcal{C}^\infty(X, \mathcal{A}^{1,1}_{X,\mathbb{R}})$ be a smooth, real d-closed $(1,1)$-form. Assume that $[\omega] \in H^2(X, \mathbb{R})$ is an integral class. Then there exists a hermitian line bundle (L, h) on X such that $\Theta_h(L) = \omega$. In particular, $[\omega] = c_1(L)$.*

Proof. We are going to give two proofs; one which is more conceptual and one which a bit more hands on.

Proof 1. Set $\alpha = [\omega] \in H^2(X, \mathbb{R}) \subset H^2(X, \mathbb{C})$ and write $\alpha = j(\tilde{\alpha})$. Since $\mathbb{Z} \to \mathcal{O}_X$ factors through $\mathbb{Z} \to \mathbb{C} \to \mathcal{O}_X$, the map $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$ factors though $H^2(X, \mathbb{C})$. Now the exponential exact sequence yields an exact sequence

$$\mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$$

so that

$$\mathrm{Im}(\mathrm{Pic}(X) \to H^2(X, \mathbb{Z})) = \mathrm{Ker}(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}) \to H^2(X, \mathcal{O}_X)).$$

Recall from Lemma 5.1 that

$$\mathrm{Ker}(H^2(X, \mathbb{C}) \to H^2(X, \mathcal{O}_X)) \cap H^2(X, \mathbb{R}) = H^{1,1}(X) \cap H^2(X, \mathbb{R})$$

and $\alpha = j(\tilde{\alpha})$ belongs to the latter space. This shows that $\tilde{\alpha} \in \mathrm{Im}(\mathrm{Pic}(X) \to H^2(X, \mathbb{Z}))$. The conclusion now follows from Lemma 3.8.

Proof 2. Cover X by polydisks U_α such that the double overlaps $U_{\alpha\beta}$ are simply connected. One can write $\omega|_{U_\alpha} = dv_\alpha$ for some real 1-form $v_\alpha = v_\alpha^{1,0} + v_\alpha^{0,1}$, hence $\omega|_{U_\alpha} = \partial v_\alpha^{0,1} + \bar{\partial} v_\alpha^{1,0}$ and $\bar{\partial} v_\alpha^{0,1} = \omega|_{U_\alpha}^{0,2} = 0$ so that $v_\alpha^{0,1} = \bar{\partial} u_\alpha$ for some function u_α by Dolbeault-Grothendieck lemma. Set $\phi_\alpha := i(\bar{u}_\alpha - u_\alpha) \in \mathcal{C}^\infty(U_\alpha, \mathbb{R})$ so that

$$\omega|_{U_\alpha} = \frac{i}{2\pi} \partial\bar{\partial}\phi_\alpha$$

Define $d^c = \frac{i}{4\pi}(\bar{\partial} - \partial)$ so that $\omega|_{U_\alpha} = dd^c\phi_\alpha$. The class $[\omega]$ corresponds in $\check{H}^2(X, \underline{\mathbb{R}})$ to the class of the 2-cocycle $(f_{\alpha\beta\gamma})$ with real values defined by $f_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma}$ and where $f_{\alpha\beta}$ is any function such that $d^c(\phi_\alpha - \phi_b) = df_{\alpha\beta}$. By assumption, there exists an integral 2-cocyle $(n_{\alpha\beta\gamma})$ and a real 1-cocycle $u = (u_{\alpha\beta})$ such that

$$f_{\alpha\beta\gamma} = n_{\alpha\beta\gamma} + (\delta u)_{\alpha\beta\gamma}$$

where δ is the Čech differential.

Now, on the overlap $U_{\alpha\beta}$, we have $\partial\bar{\partial}(\phi_\alpha - \phi_\beta) = 0$ so there exists a holomorphic function $g_{\alpha\beta} \in \mathcal{O}_X(U_{\alpha\beta})$ such that $2\mathrm{Re}(g_{\alpha\beta}) = \phi_\alpha - \phi_\beta$ by Lemma 5.3 below. This implies that $d^c(\phi_a - \phi_b) = \mathrm{Re}(d^c g_{\alpha\beta}) = \frac{1}{\pi}\mathrm{Im}(dg_{\alpha\beta})$. By what was said above, there exists $u_{\alpha\beta}$ a real 1-cycle such that if one sets $g'_{\alpha\beta} := \frac{1}{\pi} g_{\alpha\beta} - iu_{\alpha\beta}$, then $\mathrm{Im}(g'_{\alpha\beta\gamma}) \in \mathbb{Z}$. Set $\widehat{g}_{\alpha\beta} := e^{\pi g'_{\alpha\beta}} = e^{g_{\alpha\beta} - \pi i u_{\alpha\beta}}$; it defines an element in $H^1(X, \mathcal{O}_X^*)$ (hence a line bundle L) thanks to that last integrality property. Moreover, the functions ϕ_α define a metric h on L (by the relation $|e_\alpha|_h^2 = e^{-\phi_\alpha}$) since $\phi_\alpha - \phi_\beta = 2\mathrm{Re}(g_{\alpha\beta}) = \log|\widehat{g}_{\alpha\beta}|^2$ and also, one has $\Theta_h(L) = dd^c\phi_\alpha = \omega$. □

Lemma 5.3. *Let $U \subset \mathbb{C}^n$ be a connected open subset such that $H^1_{\mathrm{dR}}(U, \mathbb{R}) = 0$. Then for any real-valued function $u \in \mathcal{C}^\infty(U)$ such that $\partial\bar{\partial}u = 0$, there exists a holomorphic function f on U such that $u = \mathrm{Re}(f)$.*

Proof. Since $d\partial u = -\partial\bar{\partial}u = 0$ and the first de Rham cohomology group of U vanishes, there exists a smooth function g such that $dg = \partial u$. In particular, we have $\bar{\partial}g = 0$ hence g is holomorphic. Then

$$d(2\mathrm{Re}(g) - u) = d(g + \bar{g} - u) = \partial(g - u) + \overline{\partial(g - u)} = 0$$

Therefore there exists constant $C \in \mathbb{R}$ such that $u = 2\mathrm{Re}(g) + C$; set $f := 2g + C$. □

5.2 The Néron-Severi group

Definition 5.4 (Néron-Severi group)**.** Let X be a compact Kähler manifold. The Néron-Severi group of X, denoted by $\mathrm{NS}(X)$, is defined as the image of the Chern class morphism

$$\mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{R}).$$

Thanks to Lefschetz theorem, we have $\mathrm{NS}(X) = H^{1,1}(X) \cap \mathrm{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}))$.

The Néron-Severi group $\mathrm{NS}(X)$ is a finitely generated abelian subgroup of $H^2(X,\mathbb{R})$, hence torsion-free, of the form $\mathrm{NS}(X) = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ for some $\alpha_i = c_1(L_i)$. The rank of $\mathrm{NS}(X)$, i.e. $|I|$, is called the Picard rank of X and is usually denoted by $\rho(X)$. One has the obvious inequality $\rho(X) \leqslant h^{1,1}$ where $h^{1,1} = \dim_{\mathbb{C}} H^{1,1}(X) = \dim_{\mathbb{R}} H^{1,1}(X,\mathbb{R})$. We set $\mathrm{NS}(X)_{\mathbb{R}} := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ to be the vector space generated by $\mathrm{NS}(X)$; it has dimension $\rho(X)$.

Proposition 5.5 (A Kähler manifold without proper submanifolds)**.** *Let $X = \mathbb{C}^2/\Lambda$ be a complex 2-torus, where $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4$. For (e_i) "general", X does not admit any proper submanifold $C \subset X$, $\rho(X) = 0$ and $\mathcal{M}(X) = \mathbb{C}$. In particular, X is not projective algebraic.*

Proof. Arguing by contradiction, let C be a compact complex curve in X. Since X is diffeomorphic to the standard torus $\mathbb{C}^2/\mathbb{Z}^4$, we have $H_2(X,\mathbb{Z}) \simeq \mathbb{Z}^6$ with generators given by 6 cycles S_{ij}, the image in X of the planes $\mathbb{R}e_i + \mathbb{R}e_j$, $1 \leqslant i < j \leqslant 4$. Then one can decompose $[C] = \sum a_{ij} S_{ij}$ for $a_{ij} \in \mathbb{Z}$. In homology, one has $[C] \neq 0$ since $\int_C \omega > 0$ where ω is the Kähler form on X coming from the euclidean Kähler form $idz_1 \wedge d\bar{z}_1 + idz_2 \wedge d\bar{z}_2$ on $\mathbb{C}^2$ which is invariant under Λ.

Now, let η be the holomorphic 2-form on X coming from the Λ-invariant form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$. Write $e_i = (\alpha_i, \beta_i) \in \mathbb{C}^2$. The inclusion map $S_{ij} \hookrightarrow X$ can be read as a map $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}^2/\Lambda$ sending (u,v) to $(u\alpha_i + v\alpha_j, u\beta_i + v\beta_j)$ so that $f^*\eta = (\alpha_i\beta_j - \alpha_j\beta_i) du \wedge dv$ hence $\int_{S_{ij}} \eta = \alpha_i\beta_j - \alpha_j\beta_i$. Moreover, one has $\int_C \eta = 0$ for degree reason, so we get the relation $\sum_{i,j} a_{ij}(\alpha_i\beta_j - \alpha_j\beta_i) = 0$.

Now, if we choose the α_i, β_i in such a way that $\alpha_i\beta_j$ are linearly independent over $\mathbb{Z}$, we get a contradiction. It is easy to construct such numbers α_i, β_j explicitly. Alternatively, for each $(a_{ij}) \in \mathbb{Z}^6$, the equation $\sum_{i,j} a_{ij}(\alpha_i\beta_j - \alpha_j\beta_i) = 0$ defines a hypersurface in $H_a \subset \mathbb{C}^8$ and it is enough to choose $(\alpha, \beta) \in (\mathbb{C}^2)^4 \setminus \cup_{a \in \mathbb{Z}^6 \setminus 0} H_a$ linearly independent over $\mathbb{C}$ (an open condition).

The claim on the Picard rank of X follows easily from the argument above. Indeed, we have showed more generally that any class $[c] \in H_2(X,\mathbb{Q})$ such that $\int_c \eta = 0$ must be zero. We claim that $H^2(X,\mathbb{Q}) \cap H^{1,1}(X) = 0$, which clearly implies that $\rho(X) = 0$. Indeed, let $[\alpha] \in H^2(X,\mathbb{Q}) \cap H^{1,1}(X)$, it induces a unique linear form $\Phi_\alpha \in H^2(X,\mathbb{Q})^\vee$ defined by $[\gamma] \mapsto \int_X \alpha \wedge \gamma$. By the universal coefficient theorem, the map $\delta : H_2(X,\mathbb{Q}) \to H^2(X,\mathbb{Q})^\vee$ sending $[c]$ to $[c]\cap$ is an isomorphism. Therefore, there exists $[c_\alpha] \in H_2(X,\mathbb{Q})$ such that $\delta([c_\alpha]) = \Phi_\alpha$. In other words, we have

$$\forall [\gamma] \in H^2(X,\mathbb{Q}), \quad \Phi_\alpha([\gamma]) = \int_X \alpha \wedge \gamma = [c_\alpha] \cap [\gamma] = \int_{c_\alpha} \gamma.$$

Applying this to $\gamma = \eta$, we find $\int_{c_\alpha} \eta = 0$, hence $[c_\alpha] = 0$ and $[\alpha] = 0$.

Finally, if X had a non-constant meromorphic function f, then the support of $\mathrm{div}(f)$ would yield compact curves inside X, but the existence of those has been ruled out already. $\square$

Remark 5.6 (On the Picard rank)**.** The vector space $\mathrm{NS}(X)_{\mathbb{R}} \subset H^{1,1}(X, \mathbb{R})$ can be very small even though $h^{1,1}$ is large. For instance, a "general" torus $X = \mathbb{C}^n/\Lambda$ satisfies $\rho(X) = 0$, while $h^{1,1}(X) = \frac{n(n-1)}{2}$, cf Proposition 5.5. Such an example can of course never be a projective manifold (for which $\rho(X) \geqslant 1$, thanks to the existence of an ample line bundle arising as the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ under an embedding $X \hookrightarrow \mathbb{P}^N$).

The gap phenomenon $\rho < h^{1,1}$ may still occur if X is projective, but of course in that case one has $\rho \geqslant 1$. Examples can already be found dimension two, as there exist projective surfaces (called $K3$ surfaces) for which $\rho(X) = 1$ but $h^{1,1}(X) = 20$.

5.3 The Picard and Albanese varieties

5.3.1 The Picard variety

Definition 5.7. Let X be a compact Kähler manifold. We set

$$\mathrm{Pic}^\circ(X) := \mathrm{Ker}\big(\mathrm{Pic}(X) \to H^2(X, \mathbb{Z})\big).$$

In particular, a line bundle L such that $c_1(L) = 0$ belongs to $\mathrm{Pic}^\circ(X)$, but the converse need not be true since $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ is not injective in general (essentially because $H^2(X, \mathbb{Z})$ may have torsion).

Since the exponential map induces a surjection $H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^*)$, the exponential exact sequence yields a short exact sequence

$$0 \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Pic}^\circ(X) \longrightarrow 0. \tag{5.28}$$

Proposition 5.8. *The group* $\mathrm{Pic}^\circ(X)$ *is naturally isomorphic to a complex torus of dimension* $b_1(X)$.

Proof. First, it follows from (5.28) that $H^1(X, \mathbb{Z})$ has no torsion. By the universal coefficient theorem, we have

$$H^1(X, \mathbb{R}) = H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

In particular, the map $H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{R})$ induced by the inclusion of sheaves $\mathbb{Z} \subset \mathbb{R}$ is injective and its image generates $H^1(X, \mathbb{R})$.

Next, let us consider at the decomposition $H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$ and the duality $H^{1,0}(X) = \overline{H^{0,1}(X)}$. In restriction to $H^1(X, \mathbb{R})$, the projection map

$$H^1(X, \mathbb{R}) \to H^{0,1}(X)$$

is an isomorphism. Thanks to Lemma 5.1, the map

$$H^1(X, \mathbb{R}) \to H^1(X, \mathcal{O}_X)$$

induced by the sheaf injection $\mathbb{R} \subset \mathcal{O}_X$ is an isomorphism. In particular, $H^1(X, \mathbb{Z}) \subset H^1(X, \mathcal{O}_X)$ has discrete image and generates $H^1(X, \mathcal{O}_X)$, i.e. it is a lattice. The proposition now follows from (5.28). $\square$

Let us now discuss a little bit the case of torsion line bundles. These are bundles $L \in \mathrm{Pic}(X)$ such that there exists an integer $m \geqslant 1$ such that $L^{\otimes m} \simeq \mathcal{O}_X$. Now, et us recall the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}^\circ(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "c_1^{\mathbb{Z}}"] \arrow[dr, "c_1"'] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d, "j"] & H^2(X,\mathcal{O}_X) \\ & & & H^2(X,\mathbb{R}) & \end{tikzcd}$$

where the top row is exact. Write $H^2(X,\mathbb{Z}) = \mathbb{Z}^r \oplus G$ where $r = b_2(X)$ and G is torsion (abelian finite). If L is torsion, then $c_1(L) = 0 \in H^2(X,\mathbb{R})$. If $c_1^{\mathbb{Z}}(L) = 0$, then $L \in \mathrm{Pic}^\circ(X)$.

Otherwise, $c_1^{\mathbb{Z}}(L) = (0,g)$ for some $g \in G$ non trivial. Let us study the converse. Since $\{0\} \times G$ is in the kernel of $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$, any element of the form $\gamma = (0,g) \in H^2(X,\mathbb{Z})$ lies in the image of $c_1^{\mathbb{Z}}$. That is, there exists $L_\gamma \in \mathrm{Pic}(X)$ such that $c_1^{\mathbb{Z}}(L_\gamma) = \gamma$. Therefore, $L_\gamma^{\otimes m} \in \mathrm{Pic}^\circ(X)$ for $m = |G|$. Since $\mathrm{Pic}^\circ(X)$ is a complex torus, an element in $\mathrm{Pic}^\circ(X)$ has a root of any given order. Therefore, up to twisting L_γ by an element of $\mathrm{Pic}^\circ(X)$, one can always achieve that L_γ is torsion and $c_1^{\mathbb{Z}}(L_\gamma) = \gamma$. Therefore, we get an exact sequence

$$0 \longrightarrow \mathrm{Pic}^\circ(X)_{\mathrm{tor}} \longrightarrow \mathrm{Pic}(X)_{\mathrm{tor}} \longrightarrow H^2(X,\mathbb{Z})_{\mathrm{tor}} \longrightarrow 0.$$

Note that the universal coefficient theorem enables to identify $H^2(X,\mathbb{Z})_{\mathrm{tor}}$ with $H_1(X,\mathbb{Z})_{\mathrm{tor}}$.

5.3.2 The Albanese variety

Let us consider the map

$$\begin{array}{rccc} \psi: & H_1(X,\mathbb{Z}) & \longrightarrow & H^0(X,\Omega_X^1)^* \\ & [\gamma] & \mapsto & \left(\omega \mapsto \int_\gamma \omega\right) \end{array}$$

where a homology class $[\gamma]$ is represented by a closed path γ. It is well-defined by Stokes formula since any holomorphic form on a compact Kähler manifold is d-closed. This map need not be injective because of potential torsion in $H_1(X,\mathbb{Z})$.

Proposition 5.9. *Given a compact Kähler manifold X, the quotient*

$$\mathrm{Alb}(X) := H^0(X,\Omega_X^1)^*/\mathrm{Im}(\psi)$$

is a complex torus of dimension $b_1(X)$. It is called the Albanese variety of X.

Proof. First, we observe that ψ factors as $H_1(X,\mathbb{Z}) \to H_1(X,\mathbb{C}) \to H^{1,0}(X)^*$. Next, recall that Serre duality identifies $H^{1,0}(X)^*$ with $H^{n-1,n}(X) = H^n(X,\Omega_X^{n-1})$ while Poincaré duality shows that the cap product with the fundamental class $[X]$ yields an isomorphism $H^{2n-1}(X,\mathbb{C}) \to H_1(X,\mathbb{C})$. All in all, we get a diagram

$$\begin{tikzcd} H_1(X,\mathbb{Z}) \arrow[r, "\psi"] \arrow[dr] & H^{1,0}(X)^* & H^{n-1,n}(X) \arrow[l, "\cong"'] \\ & H_1(X,\mathbb{C}) \arrow[u] & H^{2n-1}(X,\mathbb{C}) \arrow[l, "\cong", "{[X]\cap}"'] \arrow[u] \end{tikzcd}$$

where the rightmost vertical arrow is the projection induced by the Hodge decomposition under the isomorphism (2.14) between de Rham and singular cohomology. We claim that the diagram is commutative. Indeed, let $[\gamma] \in H_1(X, \mathbb{C})$ which we realize as the cap product of $[X]$ and $[\alpha] \in H^{2n-1}(X, \mathbb{C})$. We view α as a d-closed form of degree $2n-1$. Without loss of generality, one can assume that α is harmonic with respect to some fixed Kähler metric on X. We need to show that for any $\omega \in H^{1,0}(X)$, we have

$$\psi([\gamma])(\omega) = \int_\gamma \omega = \int_X \alpha_{n-1,n} \wedge \omega.$$

Now since ω is $(1,0)$, one has

$$\int_X \alpha_{n-1,n} \wedge \omega = \int_X \alpha \wedge \omega = [X] \cap ([\alpha] \cup [\omega]) = ([X] \cap [\alpha]) \cap [\omega] = [\gamma] \cap [\omega]$$

and the latter does indeed coincide with $\int_\gamma \omega$.

Poincaré duality works over $\mathbb{Z}$ so that we have another commutative diagram

$$\begin{tikzcd}
H^{1,0}(X)^* & H^{n-1,n}(X) \arrow[l, "\cong"'] \\
H_1(X, \mathbb{C}) \arrow[u] & H^{2n-1}(X, \mathbb{C}) \arrow[l, "\mathrm{PD}"'] \arrow[u] \\
H_1(X, \mathbb{Z}) \arrow[u] \arrow[uu, bend left, "\psi"] & H^{2n-1}(X, \mathbb{Z}) \arrow[l, "\mathrm{PD}"'] \arrow[u] \arrow[uu, bend right, "\hat{\psi}"']
\end{tikzcd}$$

By the universal coefficient theorem, $H^{2n-1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = H_{2n-1}(X, \mathbb{R})$. Therefore, the image $\mathrm{Im}(H^{2n-1}(X, \mathbb{Z}) \to H^{2n-1}(X, \mathbb{R}))$ is a lattice. Since

$$H^{2n-1}(X, \mathbb{C}) = H^{n-1,n}(X) \oplus \overline{H^{n-1,n}(X)},$$

the projection map

$$H^{2n-1}(X, \mathbb{R}) \to H^{n-1,n}(X)$$

is an isomorphism. We infer that the image of

$$\hat{\psi} : H^{2n-1}(X, \mathbb{Z}) \to H^{n,n-1}(X)$$

is a lattice, hence so is $\mathrm{Im}(\psi)$. This ends the proof of the proposition. $\square$

Remark 5.10. The proof shows more generally that $\mathrm{Alb}(X)$ is isomorphic to the torus

$$H^{n-1,n}(X) / \mathrm{Im}(H^{2n-1}(X, \mathbb{Z}) \to H^{2n-1}(X, \mathbb{C}) \to H^{n-1,n}(X)).$$

In particular, if X is a curve ($n = 1$), then $\mathrm{Alb}(X)$ is isomorphic to $\mathrm{Pic}^\circ(X)$.

Given a fixed basepoint $x_0 \in X$, we define

$$\begin{array}{rccc}
\mathrm{alb} : & X & \longrightarrow & \mathrm{Alb}(X) \\
& x & \mapsto & H^0(X, \Omega_X^1) \ni \omega \mapsto \int_{x_0}^x \omega
\end{array}$$

A few comments are in order.

• The integral is taken over any path from x_0 to x. If two different paths are chosen, the integrals $\int_{x_0}^{x} \omega$ will differ by a complex number of the form $\int_\gamma \omega$ where γ is a loop at x_0. The latter integral only depends on the homotopy class of γ (if γ_1 and γ_2 are homotopy equivalent, then $\gamma_1 - \gamma_2 = \partial S$ for some 2-chain S), and only the class of γ under the map $\pi_1(X) \to H_1(X,\mathbb{Z})$. Therefore $\mathrm{alb}(x)$ is well-defined as an element in $H^0(X,\Omega^1_X)^*/H_1(X,\mathbb{Z})$.

• The choice of a different basepoint x_0 results in a translation in the torus $\mathrm{Alb}(X)$.

Proposition 5.11. *The map* alb *is holomorphic and its differential induces an isomorphism*

$$\begin{array}{ccc} H^0(\mathrm{Alb}(X),\Omega^1_{\mathrm{Alb}(X)}) & \xrightarrow{\simeq} & H^0(X,\Omega^1_X) \\ \alpha & \mapsto & \mathrm{alb}^*\alpha \end{array}$$

on the space of global holomorphic 1*-forms.*

Proof. Let us start with the first assertion, which can be checked locally in a coordinate chart. Then $\omega = \sum f_i(x)dz_i$ and one can choose $\gamma(t) = tx$ for $t \in [0,1]$. One then has to see that $x \mapsto \int_0^1 f_i(tx)x_i dt$ is holomorphic, which is clear.

For the second assertion, since both spaces have the same dimension, it suffices to show injectivity. It is convenient to fix a basis $\{\omega_1,\dots,\omega_k\}$ of $H^0(X,\Omega^1_X)$. The space

$$\Lambda := \Big\{ \Big(\int_\gamma \omega_i \Big)_{i=1\dots k}, \gamma \in H_1(X,\mathbb{Z}) \Big\} \subset \mathbb{C}^k$$

forms a lattice in $\mathbb{C}^k$ and under the natural identifications one can write

$$\mathrm{alb}(x) = \Big(\int_{x_0}^{x} \omega_1, \dots, \int_{x_0}^{x} \omega_k \Big) \in \mathbb{C}^k/\Lambda.$$

An elementary computation shows that

$$\mathrm{alb}^*(dz_i)(v) = \omega_i(v) \tag{5.29}$$

for any $x \in X$ and $v \in T_{X,x}$. The sought injectivity follows. □

Remark 5.12. Let $X = V/\Gamma$ be a complex torus. The universal cover $\pi : V \to X$ allows to identify $T_{X,0}$ with V. Moreover, we have $H^0(X,\Omega^1_X) \simeq V^*$ so that $H^0(X,\Omega^1_X)^* \simeq V^{**} \simeq V$. Next, under the identification $H_1(X,\mathbb{Z}) \simeq \Gamma$ the embedding $H_1(X,\mathbb{Z}) \to H^0(X,\Omega^1_X)^*$ is given by the inclusion $\Gamma \subset V \simeq V^{**}$. Choosing the basepoint $x_0 = 0$ the Albanese map $X \to \mathrm{Alb}(X) = V/\Gamma$ is the identity.

One way to see this is as follows. Choosing a basis allows one to identify $V \simeq \mathbb{C}^k$. If $x = [(x_i)] \in \mathbb{C}^k/\Gamma$, we have

$$\mathrm{alb}(x) = \Big[\Big(\int_0^x dz_1, \dots, \int_0^x dz_k \Big) \Big] = [(x_i)] = x,$$

as claimed.

Proposition 5.13. *Let X be a compact Kähler manifold, fix $x_0 \in X$ that we use to define* alb*. Let $f : X \to T$ be a holomorphic map to a complex torus such that $f(x_0) = 0$. There exists a unique morphism of complex tori $g : \mathrm{Alb}(X) \to T$ such that $f = g \circ \mathrm{alb}$.*

Proof. Let us start with existence. Write $T = W/\Gamma$. We also set $V := H^{1,0}(X)$ and $\Lambda = \mathrm{Im}(H_1(X, \mathbb{Z}) \to V^*)$. The differential of f induces a pullback map $W^* \simeq H^0(T, \Omega^1_T) \to H^0(X, \Omega^1_X) = V$ hence a map $\tilde{g} : V^* \to W^{**} \simeq W$. Let $\pi : \widetilde{X} \to X$ be the universal cover of X. We have a diagram

$$\begin{tikzcd} & V^* \arrow[d, "\tilde{g}"] \\ \widetilde{X} \arrow[ur, "\widetilde{\mathrm{alb}}"] \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & W \arrow[d] \\ X \arrow[r, "f"] & W/\Gamma \end{tikzcd}$$

Here, we pick a point $\tilde{x}_0 \in \widetilde{X}$ above x_0 and require that the lifts $\widetilde{\mathrm{alb}}$ and $\tilde{f}$ send $\tilde{x}_0$ to 0. We claim that

$$\tilde{g} \circ \widetilde{\mathrm{alb}} = \tilde{f}. \tag{5.30}$$

More precisely, we claim that for any $\tilde{x} \in \widetilde{X}$, the element $\mathrm{ev}_{\tilde{f}(\tilde{x})} \in W^{**}$ coincides with $\tilde{g}(\widetilde{\mathrm{alb}}(\tilde{x}))$. . Given $v \in V$, we have $\widetilde{\mathrm{alb}}(\tilde{x})(v) = \int_{\tilde{x}_0}^{\tilde{x}} \pi^* v$. Therefore, we have for any $\tau \in W^*$

$$\tilde{g}(\widetilde{\mathrm{alb}}(\tilde{x}))(\tau) = \int_{\tilde{x}_0}^{\tilde{x}} \tilde{f}^* \tau = \int_0^{\tilde{f}(\tilde{x})} \tau = \tau(\tilde{f}(\tilde{x}))$$

where we see τ respectively as a constant 1-form on W or an element in the dual of W. This shows (5.30).

Finally, we need to show $\tilde{g}$ induces a map $V^*/\Lambda \to W/\Gamma$. Equivalently, one needs to check that $\tilde{g}(\Lambda) \subset \Gamma$. This is a consequence of the formula

$$\int_\gamma f^* \tau = \int_{f_* \gamma} \tau,$$

where γ is any closed path on X and $\tau \in W^*$ is any global 1-form on T. This establishes the existence of g.

As for uniqueness, it follows from Lemma 5.14 below. Indeed, given $y \in \mathrm{Alb}(X)$, choose a decomposition $y = \sum_j \mathrm{alb}(x_j)$ for some points $x_j \in X$. Then one must have $g(y) = \sum_j g(\mathrm{alb}(x_j)) = \sum_j f(x_j)$ so that f determines g uniquely (once existence of g is established). □

Lemma 5.14. *The image* $\mathrm{alb}(X) \subset \mathrm{Alb}(X)$ *generates* $\mathrm{Alb}(X)$ *as a group.*

Proof. It is enough to show that the map

$$a_m : X^m \to \mathrm{Alb}(X), (x_1, \dots, x_m) \mapsto \sum_{j=1}^m \mathrm{alb}(x_j)$$

is a submersion at one point, for some m large enough. We first observe that if we pick enough points x_i, then an element $\omega \in V := H^{1,0}(X)$ such that $\omega_{x_i} = 0$ for all i has to vanish identically. Indeed, let us consider the kernel of the evaluation map

$$K_{x_1,\ldots,x_m} = \mathrm{Ker}\Big(V \to \bigoplus_{i=1}^{m} \Omega^1_{X,x_i}\Big).$$

One can certainly choose x_1 such that $K_{x_1} \subset V$ is a proper subspace. If $K_{x_1} \neq 0$, pick $\omega \in K_{x_1} \setminus \{0\}$ and pick x_2 such that $\omega_{x_2} \neq 0$. This yields a proper subspace $K_{x_1,x_2} \subsetneq K_{x_1}$. Eventually the process stops and

$$K_{x_1,\ldots,x_m} = 0. \tag{5.31}$$

Given $\underline{x} = (x_1, \ldots, x_m) \in X^m$ and $v = (v_1, \ldots, v_n) \in T_{X^m,\underline{x}}$, one has seen in (5.29) that

$$da_m(v) = \Big(V \ni \omega \mapsto \sum_{i=1}^{m} \omega_{x_i}(v_i)\Big).$$

To show that da_m is surjective at $\underline{x}$, we prove that is any linear form on V^* vanishing on the image of da_m must be zero. So we pick $\omega \in V \simeq V^{**}$ such that $\sum_{i=1}^{m} \omega_{x_i}(v_i) = 0$ for all v. This implies that $\omega_{x_i} = 0$ for all i, hence $\omega = 0$ by our choice of x_i, cf (5.31). This ends the proof of the lemma. □

5.3.3 The Abel-Jacobi map

In this paragraph, we will assume that $\dim X = 1$, i.e. X is a compact Riemann surface. We fix a point $x_0 \in X$. We have seen in Remark 5.10 that

$$\mathrm{Pic}^\circ(X) \simeq \mathrm{Alb}(X) \tag{5.32}$$

in that case. More precisely, consider the diagram

$$\begin{tikzcd} H^1(X, \mathcal{O}_X) \arrow[r, "\Phi"] & H^{1,0}(X)^* \\ H^1(X, \mathbb{Z}) \arrow[u] \arrow[r] & H_1(X, \mathbb{Z}) \arrow[u] \end{tikzcd}$$

where Φ is the composition of the isomorphism $H^1(X, \mathcal{O}_X) \to H^{0,1}(X)$ with the map $H^{0,1}(X) \to H^{1,0}(X)^*, \eta \mapsto (\omega \mapsto \int_X \eta \wedge \omega)$, the vertical arrows are induced by the exponential exact sequence and ψ respectively, and the bottom horizontal arrow is the Poincaré duality $\alpha \mapsto [X] \cap \alpha$; it is an isomorphism. We claim that the diagram is commutative, which will imply (5.32). Indeed, if $\alpha \in H^1(X, \mathbb{Z})$, we view α as a d-closed 1-form and we need to check that $\int \alpha_{0,1} \wedge \omega = \langle \omega, [X] \cap \alpha \rangle$ for any $\omega \in H^{1,0}(X)$. But this follows from $\int \alpha_{0,1} \wedge \omega = \int \alpha \wedge \omega$ and the basic properties of cap and cup products.

For $x \in X$, we look at the degree zero divisor

$$D = x - x_0$$

with associated line bundle $L := \mathcal{O}_X(D)$. Since X is a curve, Poincaré duality shows that $H^2(X,\mathbb{Z}) = \mathbb{Z}$ and the map $\mathrm{Pic}(X) \to H^2(X,\mathbb{Z})$ induced by the exponential exact sequence is simply the degree map one divisors are identified with line bundles. In particular, $\mathrm{Pic}^\circ(X)$ corresponds to degree zero divisors.

Proposition 5.15. *Under the isomorphism (5.32), we have $\mathcal{O}_X(x - x_0) = \mathrm{alb}(x)$.*

Remark 5.16. A similar result holds in higher dimension for divisors cohomologous to zero, but one has to replace the Albanese variety (and map) by a different Hodge theoretic object, cf [Voi07, Proposition 12.7].

Proof. The fact that $L \in \mathrm{Pic}(X)$ maps to zero in $H^2(X,\mathbb{Z})$ in the exponential exact sequence means that one can find a cover (U_i) of X trivializing L and such that the transition functions g_{ij} of L satisfy

$$g_{ij} = e^{2\pi i f_{ij}}, \quad f_{ij} \in \mathcal{O}_X(U_{ij}) \quad \text{and} \quad f_{ij} + f_{jk} + f_{ki} = 0 \text{ on } U_{ijk}.$$

Up to refining the covering, one can find $f_i \in \mathcal{C}^\infty(U_i)$ such that $f_{ij} = f_i - f_j$ since $H^1(X, \mathcal{C}^\infty_X) = 0$, cf Proposition (1.3). Consider the $(0,1)$-forms

$$\alpha_i = -\bar{\partial} f_i \in \mathcal{A}^{0,1}(U_i).$$

Since f_{ij} is holomorphic, the α_i glue to a global ($\bar{\partial}$-closed) $(0,1)$-form α. It is not difficult to check that under the isomorphism $H^1(X,\mathcal{O}_X) \simeq H^{0,1}(X)$, the image of α in $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$ coincides with the class of $\mathcal{O}_X(x - x_0)$. Therefore, in order to prove the proposition it is enough to show that

$$\int_X \alpha \wedge \omega = \int_{x_0}^{x} \omega, \quad \text{for any} \quad \omega \in H^0(X, \Omega^1_X). \tag{5.33}$$

In order to do that, we first show that L is $\mathcal{C}^\infty$-trivial and D is $\mathcal{C}^\infty$-linearly equivalent to zero. One could argue abstractly by saying that the map $H^1(X, (\mathcal{C}^\infty_X)^*) \to H^2(X,\mathbb{Z})$ is an isomorphism. Or, in a more down to earth way, define $g_i := e^{2\pi i f_i} \in \mathcal{C}^\infty(U_i)$ and observe that $g_i = g_{ij} g_i$ so that the (g_i) glue to a smooth trivializing section τ of L. Let σ be a meromorphic section of L such that $\mathrm{div}(\sigma) = D$ and set

$$\chi := \frac{\sigma}{\tau} : X \to \mathbb{P}^1.$$

The function χ is smooth, and takes values in $\mathbb{C}^*$ in restriction to $X \setminus \{x_0, x\}$. Moreover, we have

$$\alpha = \frac{1}{2\pi i} \bar{\partial} \log \chi \quad \text{on} \quad X \setminus \{x_0, x\}.$$

Of course $\log \chi$ is multi-valued on $X \setminus \{x_0, x\}$. Consider a smooth path $\gamma : [0,1] \to \mathbb{P}^1$ such that $\gamma(0) = 0, \gamma(1) = \infty$. Using the parametric transversality theorem, one can show that if γ is generic, then $\Gamma := \chi^{-1}(\gamma)$ is a smooth manifold with boundary such that $\partial\Gamma = D$. For $\varepsilon > 0$, we denote by Γ_ε an open neighborhood if $\Gamma \subset X$ of size ε with smooth boundary. On $X \setminus \Gamma$, one has

$$\alpha \wedge \omega = \frac{1}{2\pi i} d(\log \chi \cdot \omega)$$

for degree reasons. Then

$$\int_X \alpha \wedge \omega = \int_{X \setminus \Gamma} \alpha \wedge \omega = \lim_{\varepsilon \to 0} \int_{X \setminus \Gamma_\varepsilon} \alpha \wedge \omega = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{\partial \Gamma_\varepsilon} \log \chi \cdot \omega.$$

As $\varepsilon \to 0$, the boundary $\partial \Gamma_\varepsilon$ projects via χ to two very close paths $\gamma_\varepsilon^\pm$ almost connecting 0 to ∞ (resp. ∞ to 0). Moreover, we have

$$\lim_{\varepsilon \to 0} \log \chi|_{\gamma_\varepsilon^+} = \lim_{\varepsilon \to 0} \log \chi|_{\gamma_\varepsilon^-} - 2\pi i.$$

This shows that

$$\int_X \alpha \wedge \omega = \int_\gamma \omega = \int_{x_0}^x \omega,$$

which proves the proposition. □

Corollary 5.17. *If X has genus at least one, then the Albanese map $\mathrm{alb} : X \to \mathrm{Alb}(X)$ is injective.*

Proof. By Proposition 5.15, it is enough to prove that $\mathcal{O}_X(x - x_0)$ is isomorphic to $\mathcal{O}_X(y - x_0)$ if and only if $x = y$. But the former happens iff $x - y$ is a principal divisor $\operatorname{div} f$ where $f \in \mathcal{M}(X)$. Then f would yield a map $X \to \mathbb{P}^1$ with degree one, hence an isomorphism. □

5.4 The Hard Lefschetz theorem

Let (X, ω) be a Kähler manifold of dimension n. The Kähler form ω is real, so it defines a linear map

$$L : \Omega^k_{X,\mathbb{R}} \to \Omega^{k+2}_{X,\mathbb{R}}, \quad \alpha \mapsto \omega \wedge \alpha.$$

Lemma 5.18. *For $k \leqslant n$, the map*

$$L^{n-k} : \Omega^k_{X,\mathbb{R}} \to \Omega^{2n-k}_{X,\mathbb{R}}$$

is an isomorphism.

Remark 5.19. In particular, $L^r : \Omega^k_{X,\mathbb{R}} \to \Omega^{k+2r}_{X,\mathbb{R}}$ is an isomorphism for any $0 \leqslant r \leqslant n - k$.

Remark 5.20. The bound is optimal as L^{n-k+1} need not be injective on $\Omega^k_{X,\mathbb{R}}$ anymore even when the target space is not trivial, i.e. $2n - k + 2 \leqslant 2n$. E.g. take $k = n$ and α a $(n, 0)$-form, then $L(\alpha + \bar{\alpha}) = 0$ for degree reasons.

Proof. Since the two vector bundles have the same rank, it is sufficient to show that for every $U \subset X$ the morphism

$$L^{n-k} : C^\infty(U, \Omega^k_{X,\mathbb{R}}) \to C^\infty(U, \Omega^{2n-k}_{X,\mathbb{R}}), \quad \alpha \mapsto \omega^{n-k} \wedge \alpha$$

is injective. Recall the commutation relation

$$[L, \Lambda]\alpha = (k - n)\alpha$$

for every $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$. By definition $[L^r, \Lambda] = L[L^{r-1}, \Lambda] + [L, \Lambda]L^{r-1}$, so we get inductively for every k-form α

$$[L^r, \Lambda]\alpha = (r(k-n) + r(r-1))L^{r-1}\alpha. \tag{5.34}$$

We will prove by induction over $k+r$ (with $k \in \{0, \dots, n\}$ and $r \in \{0, \dots, n-k\}$, so that $k+r \leqslant n$) that $L^r : \Omega^k_{X,\mathbb{R}} \to \Omega^{k+2r}_{X,\mathbb{R}}$ is injective. For $k+r \leqslant 1$ this is clear, thus let α be a k-form such that $L^r\alpha = 0$, with $k+r \geqslant 2$. We can assume $r \geqslant 1$ without loss of generality. By (5.34) this implies that

$$L^{r-1}(L\Lambda\alpha - (r(k-n) + r(r-1))\alpha) = 0, \tag{5.35}$$

so by the induction hypothesis we find

$$L\Lambda\alpha - (r(k-n) + r(r-1))\alpha = 0.$$

Therefore we get $(r(k-n) + r(r-1))\alpha = L\beta$ with $\beta = \Lambda\alpha$ of degree $k-2$. Furthermore $L^{r+1}\beta = 0$ so by the induction hypothesis, we have $\beta = 0$. Since

$$r((k-n) + r - 1) \leqslant -r \leqslant -1,$$

we infer $\alpha = 0$, thus concluding the proof of the lemma. □

Definition 5.21. Let (X, ω) be a Kähler manifold of dimension n and let $k \leqslant n$ be a non-negative integer. We say that a form $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$ is primitive if $L^{n-k+1}\alpha = 0$.

Proposition 5.22. *Let $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$ for some arbitrary $0 \leqslant k \leqslant 2n$.*

- (i) *If $k \leqslant n$, then α is primitive if and only if $\Lambda\alpha = 0$.*
- (ii) *α admits a unique decomposition of the form*

$$\alpha = \sum_r L^r \alpha_r$$

such that α_r is primitive of degree $k - 2r \leqslant \inf(2n-k, k)$.

The decomposition above is called the Lefschetz decomposition of α.

Proof. (i) Set $r := n-k+1$; by (5.34), we have

$$L^r\Lambda\alpha = \Lambda L^r\alpha.$$

If α is primitive, then $L^r\Lambda\alpha = 0$ hence Lemma 5.18 applies (since $r = n-k+1 \leqslant n - (k-2) = n - \deg(\Lambda\alpha)$) to show $\Lambda\alpha = 0$.
Conversely, if $\Lambda\alpha = 0$, then $\Lambda L^r\alpha = 0$. Recall that $\Lambda = \star^{-1}L\star$, so that $L\beta = 0$ where $\beta = \star L^r\alpha$ has degree $k-2$. Since $1 \leqslant n-(k-2)$, Lemma 5.18 shows $\beta = 0$, hence $L^r\alpha = 0$.

(ii) First of all, one can assume $k \leqslant n$. Indeed, if $k > n$, Lemma 5.18 allows us to write $\alpha = L^{k-n}\beta$ with $\deg(\beta) = 2n-k < n$ and apply the result to $\beta = \sum L^s\beta_s$ so that $\alpha = \sum L^{k-n+s}\beta_s$ and we indeed have $\deg(\beta_s) = 2n-k-2s \leqslant 2n-k = \inf(2n-k, k)$.

Next, let us show uniqueness; we assume $k \leqslant n$ for now. We assume $\sum_r L^r \alpha_r = 0$ with α_r primitive and proceed by induction on the degree k. If the smallest r appearing in the sum is non-zero (i.e. $\alpha_0 = 0$), then we have $L\big(\sum_r L^{r-1}\alpha_r\big) = 0$ with $1 \leqslant n - \deg(L^{r-1}\alpha_r) = n - k + 2$ so that Lemma 5.18 yields $\sum_r L^{r-1}\alpha_r = 0$ hence $\alpha_r = 0$ for all r by induction. It remains to see what happens if $\alpha_0 \neq 0$. Since α_0 is primitive, we have $L^{n-k+1}\alpha_0 = 0$. Hence

$$L^{n-k+1} \sum_{r>0} L^r \alpha_r = 0 = L^{n-k+2} \sum_{r>0} L^{r-1}\alpha_r,$$

so that Lemma 5.18 implies $\sum_{r>0} L^{r-1}\alpha_r = 0$, which by induction shows $\alpha_r = 0$ for all $r > 0$, hence $\alpha_0 = 0$, too.

If $k \geqslant n$, we have $r \geqslant k - n$ and $0 = L^{k-n}\big(\sum_r L^{r-(k-n)}\alpha_r\big)$. By Lemma 5.18, since $k - n \leqslant k - n + 2r = n - \deg(L^{r-(k-n)}\alpha_r)$, we find $\sum_r L^{r-(k-n)}\alpha_r = 0$. But now, $\deg(L^{r-(k-n)}\alpha_r) = 2n - k \leqslant n$ hence one can apply the first part.

It remains to show existence. As we explained above, one can assume $k \leqslant n$. The element $L^{n-k+1}\alpha$ has degree $2n - k + 2$ hence Lemma 5.18 guarantees the existence of β of degree $k - 2$ such that $L^{n-k+2}\beta = L^{n-k+1}\alpha$. The k-form $\alpha_0 := \alpha - L\beta$ is therefore primitive and $\alpha = \alpha_0 + L\beta$. It is now clear that one can argue by induction to show the existence of a Lefschetz decomposition for β, hence for α, too. □

Note now that since ω is d-closed, the linear map L induces a linear map on the de Rham cohomology groups

$$L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R}), \quad [\alpha] \mapsto [\omega \wedge \alpha]. \tag{5.36}$$

Theorem 5.23 (Hard Lefschetz theorem). *Let (X, ω) be a compact Kähler manifold of dimension n. Then for every $k \leqslant n$, the map*

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R}) \tag{5.37}$$

is an isomorphism.

Proof. Since $[\Delta_d, L] = 0$, the image $L(\alpha)$ of a harmonic form α is harmonic. Since X is compact, Hodge theory shows that it is equivalent to show that the induced morphism

$$L^{n-k} : \mathcal{H}^k(X, \mathbb{R}) \to \mathcal{H}^{2n-k}(X, \mathbb{R}) \tag{5.38}$$

is an isomorphism. By Poincaré duality, both vector spaces have the same dimension; furthermore by Lemma 5.18, the morphism

$$L^{n-k} : C^\infty(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty(X, \Omega^{2n-k}_{X,\mathbb{R}}) \tag{5.39}$$

is injective. In particular it is injective on harmonic forms, so the statement follows. □

Definition 5.24. Let (X, ω) be a compact Kähler manifold of dimension n. We say that $[\alpha] \in H^k(X, \mathbb{R})$, $k \leqslant n$ is primitive if $L^{n-k+1}[\alpha] = 0$. We denote by $H^k(X, \mathbb{R})_{\mathrm{prim}} \subset H^k(X, \mathbb{R})$ the subspace of primitive classes.

As a consequence of Proposition 5.22, we obtain

Corollary 5.25. *Let (X,ω) be a compact Kähler manifold. Then every element $[\alpha] \in H^k(X,\mathbb{R})$ admits a unique decomposition of the form*

$$[\alpha] = \sum_r L^r[\alpha_r] \tag{5.40}$$

such that $\alpha_r \in H^{k-2r}(X,\mathbb{R})_{\text{prim}}$ with $k-2r \leqslant \inf(2n-k,k)$. In particular we have

$$H^k(X,\mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X,\mathbb{R})_{\text{prim}}. \tag{5.41}$$

This Lefschetz decomposition plays an important role in the Hodge index theorem (cf. Theorem 5.30 below). The Hard Lefschetz theorem also holds for the de Rham cohomology with complex coefficients. Since the Kähler form ω is $\bar{\partial}$-closed, the Hodge decomposition shows immediately that for all $p+q \leqslant n$, we have an isomorphism

$$L^{n-p-q} : H^{p,q}(X) \to H^{n-q,n-p}(X).$$

5.5 The Hodge index theorem

We will now consider the Hodge index theorem for the intersection form on $H^2(X,\mathbb{C})$. For the sake of simplicity of notation, we will restrict ourselves to the case of a compact Kähler surface and refer to [Voi07, § 6.3.2] for an account of the full picture in arbitrary dimension. We start with a technical lemma.

Lemma 5.26. *Let $U \subset \mathbb{C}^2$ be an open subset and endow T_U with the standard metric $h = 2(dz_1 \otimes d\overline{z_1} + dz_2 \otimes d\overline{z_2})$. Let $\alpha \in C^\infty(U,\Omega_U^{p,q})$ be a primitive two-form. Then we have*

$$*\alpha = (-1)^q\alpha.$$

Proof. We will prove the claim where α is of type $(1,1)$, the other cases are analogous. Let $\omega = i(dz_1 \wedge d\overline{z_1} + dz_2 \wedge d\overline{z_2})$ be the Kähler form, and set

$$\alpha = \alpha_{1,2}dz_1 \wedge d\overline{z_2} + \alpha_{2,1}dz_2 \wedge d\overline{z_1} + \alpha_{1,1}dz_1 \wedge d\overline{z_1} + \alpha_{2,2}dz_2 \wedge d\overline{z_2}$$

where the $\alpha_{j,k}$ are differentiable functions. The volume form is

$$\frac{\omega^2}{2!} = i^2 dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2},$$

so $\alpha \wedge *\overline{\alpha} = \{\alpha,\alpha\}$ vol implies

$$*\alpha = -\alpha_{1,2}dz_1 \wedge d\overline{z_2} - \alpha_{2,1}dz_2 \wedge d\overline{z_1} + \alpha_{1,1}dz_2 \wedge d\overline{z_2} + \alpha_{2,2}dz_1 \wedge d\overline{z_1}.$$

Furthermore, $L\alpha = \omega \wedge \alpha = i(\alpha_{1,1}+\alpha_{2,2})dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2}$ equals zero if and only if $\alpha_{1,1} = -\alpha_{2,2}$. This implies the claim. □

The above result easily implies the following

Lemma 5.27. *Let (X,ω) be a Kähler manifold of dimension two, and let $\alpha \in C^\infty(X,\Omega_X^{p,q})$ be a primitive 2-form. Then we have*

$$*\alpha = (-1)^q\alpha.$$

Let now X be a compact Kähler manifold of dimension two. Then the Poincaré duality shows that we have a non-degenerate symmetric bilinear pairing

$$Q : ([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta.$$

Therefore $Q : H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \to \mathbb{R}$ defines a non-degenerate hermitian form on $H^2(X, \mathbb{C})$ via $H(\alpha, \beta) := Q(\alpha, \overline{\beta})$.

Lemma 5.28. *Let (X, ω) be a compact Kähler manifold of dimension two. The Lefschetz decomposition*

$$H^2(X, \mathbb{C}) = H^2(X, \mathbb{C})_{\text{prim}} \oplus LH^0(X, \mathbb{C}) = H^2(X, \mathbb{C})_{\text{prim}} \oplus \mathbb{C}[\omega]$$

is orthogonal.

Proof. As before we reduce the statement on cohomology to a statement on harmonic forms. By definition, a two-form α is primitive if $\omega \wedge \alpha = 0$. Therefore

$$H(\alpha, \omega) = \int_X \alpha \wedge \overline{\omega} = \int_X \alpha \wedge \omega = 0.$$

$\square$

Proposition 5.29. *The subspaces $H^{p,q}(X) \subset H^2(X, \mathbb{C})$ are orthogonal with respect to H. Furthermore $(-1)^q H$ is definite positive on the subspace*

$$H^{p,q}(X)_{\text{prim}} := H^2(X, \mathbb{C})_{\text{prim}} \cap H^{p,q}(X)$$

and we have an H-orthogonal decomposition

$$H^2(X, \mathbb{C})_{\text{prim}} = H^{2,0}(X) \oplus H^{1,1}(X)_{\text{prim}} \oplus H^{0,2}(X).$$

Proof. The orthogonality is obvious for type reasons. Note that $H^{2,0}_{\text{prim}} = H^{2,0}(X)$ and $H^{0,2}_{\text{prim}} = H^{0,2}(X)$. Let $\gamma \in H^{p,q}_{\text{prim}}$ be a non-zero class. Note also that $H^{p,q}(X) \simeq \mathcal{H}^{p,q}(X)$ and the Lefschetz operator commutes with $\Delta_{\overline{\partial}}$. Thus if α is the harmonic representative of γ, we have $\Delta_{\overline{\partial}} L\alpha = \Delta_{\overline{\partial}} \alpha = 0$ so that $L\alpha$ is the harmonic representative of $L\gamma = 0$. That is, $L\alpha = 0$ and α is primitive. Then we have by Lemma 5.27:

$$*\overline{\alpha} = (-1)^q \overline{\alpha}$$

so

$$(-1)^q H(\alpha, \alpha) = (-1)^q \int_X \alpha \wedge \overline{\alpha} = (-1)^{2q} \int_X \alpha \wedge *\overline{\alpha} = \|\alpha\|_{L^2} > 0.$$

Let us prove the last claim. If $\gamma \in H^2(X, \mathbb{C})_{\text{prim}}$, one can consider its harmonic representative α, which is primitive as we explained above. Now, consider the decomposition $\alpha = \alpha_{2,0} + \alpha_{1,1} + \alpha_{0,2}$ into types. Each $\alpha_{p,q}$ is harmonic and primitive (this is obvious for $\alpha_{2,0}$ and $\alpha_{0,2}$, hence it follows for $\alpha_{1,1}$ since $L\alpha = 0$). This gives the sought decomposition. $\square$

Theorem 5.30 (Hodge Index Theorem)**.** *Let X be a compact Kähler surface. Then the signature of the intersection form*

$$Q([\alpha],[\beta]) = \int_X \alpha \wedge \beta$$

on $H^2(X,\mathbb{R}) \cap H^{1,1}(X)$ is $(1, h^{1,1}-1)$. More generally, the signature of Q on $H^2(X,\mathbb{R})$ is $(1+2h^{2,0}, h^{1,1}-1)$.

Proof. By Lemma 5.28 and Proposition 5.29 we have a H-orthogonal decomposition

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X,\mathbb{C})_{\text{prim}} \oplus \mathbb{C}[\omega] \oplus H^{0,2}(X).$$

Now, on $H^2(X,\mathbb{R})$, H coincides with Q and we have an orthogonal decomposition

$$H^2(X,\mathbb{R}) = (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}} \oplus H^{1,1}(X,\mathbb{R})_{\text{prim}} \oplus \mathbb{R}[\omega].$$

where $H^{1,1}(X,\mathbb{R})_{\text{prim}} = H^{1,1}(X)_{\text{prim}} \cap H^2(X,\mathbb{R})$ and

$$(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}} = \{u + \bar{u}; u \in H^{2,0}\} \simeq H^{2,0}(X),$$

which has dimension $2h^{2,0}$ over $\mathbb{R}$. By Proposition 5.29, the form Q is negative definite on $H^{1,1}(X,\mathbb{R})_{\text{prim}}$. Moreover, if $u \in H^{2,0}(X)$, we have $Q(u+\bar{u}, u+\bar{u}) = 2H(u,u) > 0$ by Proposition 5.29, too. Finally, H is definite positive on $\mathbb{R}\omega$ since $\int_X \omega \wedge \overline{\omega} = \int_X \omega \wedge \omega > 0$. The theorem is proved. □

Exercise 5.31. Let (X,ω) be a compact Kähler surface, and let γ be a real closed 2-form such that $[\gamma] \in H^{1,1}(X)$. Show that

$$Q([\gamma],[\gamma]) \cdot Q([\omega],[\omega]) \leqslant Q([\gamma],[\omega])^2.$$

6 Blow-ups

6.1 Refresher on the canonical bundle

Definition 6.1. Let X be a complex manifold of dimension n. The canonical bundle of X is the line bundle $K_X := \det T_X^* = \Lambda^n \Omega_X^1$, whose sections are holomorphic n-forms.

If $(z_1, \ldots, z_n)$ is a local system of coordinates on an open set $U \subset X$, then $K_X|_U$ admits the non-vanishing section $dz_1 \wedge \ldots \wedge dz_n$.

We have the following result.

Lemma 6.2. *Let*

$$0 \longrightarrow S \xrightarrow{g} E \xrightarrow{f} Q \longrightarrow 0$$

be an exact sequence of vector bundles. Then there is an isomorphism

$$\det E \simeq \det S \otimes \det Q.$$

Proof. Set $p = \mathrm{rk}(S)$ and $q := \mathrm{rk}(Q)$. We aim to define a vector bundle map $\det S \otimes \det Q \to \det E$ which is not zero anywhere. We have two maps $\Lambda^p g : \det S \to \Lambda^p E$ and $\Lambda^q f : \Lambda^q E \to \det Q$. The first one is injective while the second one is surjective. The kernel of $\Lambda^q f$ is generated by image of the map $g \otimes \mathrm{Id}_E^{\wedge q-1} : S \otimes \Lambda^{q-1} E \to \Lambda^q E$.

At a given point $x \in X$, consider two elements $\sigma \in \det S_x, \tau \in \det Q_x$. One can write $\tau = \Lambda^q f(\hat{\tau})$ while $\hat{\tau}$ is only well-defined up to an element $\rho \in \mathrm{Im}(g \otimes \mathrm{Id}_E^{\wedge q-1})$. Since $\sigma \wedge \rho = 0$ for any such ρ, this means that the element $\sigma \wedge \hat{\tau} \in \det E$ only depends on σ and τ but not the choice of $\hat{\tau}$. This gives the desired isomorphism $\det S \otimes \det Q \to \det E$, $\sigma \otimes \tau \mapsto \sigma \wedge \hat{\tau}$. $\square$

As a consequence of Proposition 4.4 and Lemma 6.2 above, we find

Corollary 6.3. *There is an isomorphism* $K_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

Proof. The Euler exact sequence yields $K_{\mathbb{P}^n} \simeq \det \mathcal{O}(-1)^{\oplus(n+1)} \simeq \mathcal{O}(-1)^{\otimes(n+1)}$, hence the result. $\square$

6.2 Normal bundle, projective bundles

Let X be a complex manifold and let $Y \subset X$ be a smooth submanifold of codimension $k \geqslant 1$.

Definition 6.4 (Normal bundle)**.** The normal bundle $N_{Y|X}$ of $Y \subset X$ is defined as the cokernel of the map $T_Y \longrightarrow T_X|_Y$, i.e.

$$0 \longrightarrow T_Y \longrightarrow T_X|_Y \longrightarrow N_{Y|X} \longrightarrow 0.$$

The normal bundle $N_{Y|X}$ is a vector bundle on Y with rank $k = \mathrm{codim}_Y X$.

When no confusion is possible, we sometimes write N_Y instead of $N_{Y|X}$.

Proposition 6.5. *Let $Y \subset X$ be a smooth hypersurface. Then* $N_{Y|X} \simeq \mathcal{O}_Y(Y) := \mathcal{O}_X(Y)|_Y$.

Proof. Cover X with charts U_α where $Y \cap U_\alpha = (f_\alpha = 0)$. On $U_{\alpha\beta}$, there exists a non-vanishing function h such that $f_\alpha = h f_\beta$ (h is nothing but the transition function of $\mathcal{O}_X(Y)$ relative to (U_α)). We thus have $df_\alpha = h df_\beta + f_\beta dh$, hence $df_\alpha = h df_\beta$ on $Y \cap U_{\alpha\beta}$.

By dualizing the exact sequence defining the normal bundle, we get

$$0 \longrightarrow N_Y^* \longrightarrow \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0. \tag{6.42}$$

The kernel N_Y^* is of rank one, locally generated on $Y \cap U_\alpha$ by the differential $e_\alpha := df_\alpha|_Y$. This means that the cocycle $(g_{\alpha\beta} = \frac{e_\beta}{e_\alpha})_{\alpha\beta}$ associated to N_Y^* is nothing but $\frac{df_\beta}{df_\alpha}|_Y = \frac{f_\beta}{f_\alpha}|_Y = h^{-1}$, the cocyle associated to $\mathcal{O}_X(-Y)|_Y$. This proves the formula. $\square$

As a consequence of Lemma 6.2 and Proposition 6.5, we get

Corollary 6.6. *Let $Y \subset X$ be a smooth hypersurface. Then* $K_Y \simeq (K_X \otimes \mathcal{O}_X(Y))|_Y$.

In the case of complete intersections in $\mathbb{P}^n$, Corollary 6.3 and Corollary 6.6 yield

Corollary 6.7. *Let $Y \subset \mathbb{P}^n$ be a codimension k submanifold which is obtained as a complete intersection of k smooth hypersurfaces of respective degree $d_1, \ldots, d_k$. Then*

$$K_Y \simeq \mathcal{O}_{\mathbb{P}^n}(\sum_{i=1}^{k} d_i - (n+1))|_Y$$

.

As an application, we can see that the twisted cubic cannot be a complete intersection. More precisely, consider the intersection $X \subset \mathbb{P}^3$ of the three quadrics $X = (xz - y^2 = yw - z^2 = xw - yz = 0)$ in homogeneous coordinates $[x : y : z : w]$. One has

1. X is smooth, and the map $\mathbb{P}^1 \to \mathbb{P}^3$, $[u : v] \mapsto [u^3 : u^2 v : uv^2 : v^3]$ induces an isomorphism onto X.
2. X cannot be obtained as a complete intersection (otherwise its canonical bundle would be either trivial or ample, since X does not lie in any hyperplane)
3. Set-theoretically, X is the complete intersection $(xz - y^2 = z(yw - z^2) - w(xw - yz) = 0)$.

Definition 6.8 (Projective bundle)**.** Let X be a complex manifold, and let $E \to X$ be a holomorphic vector bundle of rank $r \geqslant 2$. There exists a locally trivial fiber bundle $\mathbb{P}(E) \to X$ such that $\mathbb{P}(E)_x = \mathbb{P}(E_x) \simeq \mathbb{P}^{r-1}$ is the space of complex lines of E.

Example 6.9 (Projectivized normal bundle)**.** Let $Y \subset X$ be a smooth submanifold of codimension $r \geqslant 2$. The projectivized normal bundle, $\mathbb{P}(N_{Y|X})$, is the bundle of normal directions. It fibers over Y with fibers isomorphic to $\mathbb{P}^{r-1}$.

Definition 6.10 (Tautological bundle)**.** Let $E \to X$ be a holomorphic vector bundle of rank $r \geqslant 2$, inducing $\pi : \mathbb{P}(E) \to Y$. The tautological line bundle $\tau : \mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathbb{P}(E)$ is defined by $\mathcal{O}_{\mathbb{P}(E)}(-1) := \{((x,[v]), w) \in \mathbb{P}(E) \times E_x \big| w \in \mathbb{C}v\} \subset \pi^* E = \mathbb{P}(E) \times_X E$.

The situation is described in the diagram below

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}(E)}(-1) \arrow[dr, "\tau"'] & \subset & \pi^* E \arrow[r] \arrow[d] & E \arrow[d] \\
& & \mathbb{P}(E) \arrow[r, "\pi"] & X
\end{tikzcd}$$

In particular, we have for every $x \in X$ an isomorphism

$$\mathcal{O}_{\mathbb{P}(E)}(-1)|_{\mathbb{P}(E_x)} \simeq \mathcal{O}_{\mathbb{P}(E_x)}(-1) \simeq \mathcal{O}_{\mathbb{P}^{r-1}}(-1). \tag{6.43}$$

6.3 Blow-up of a smooth submanifold

Let X be a complex manifold, and let $Y \subset X$ a complex submanifold of codimension k. Locally along Y, there exist holomorphic functions $f_1, \ldots, f_k$ with independent differentials, such that $Y = \{z | f_i(z) = 0\}$. These equations are not unique, but we have the following.

Lemma 6.11. *If $g_1, \dots, g_k$ form another system of local equations for Y, then locally in the neighbourhood of Y , there exists a matrix M_{ij} of holomorphic functions such that*

$$g_i = \sum_j M_{ji} f_j. \tag{6.44}$$

Moreover, the matrix M_{ij} is invertible along Y, and its restriction to Y is uniquely determined by the f_i, g_j.

Proof. It suffices to prove the lemma in the case where the $f_j(z)$ are the first k coordinates. The functions g_i then have the property of vanishing on $\{z | z_1 = \cdots = z_k = 0\}$. Taking the power series expansion of g_i we see immediately that we must have $g_i = \sum_{j=1}^k M_{ji} z_j$.

The fact that M_{ji} is uniquely determined can be shown by taking the differentials of (6.44) along Y, which gives the relations $dg_i = \sum_j M_{ji} df_j$. Uniqueness comes from the fact that the df_j are independent along Y. The invertibility of M_{ji} along Y, and thus in a neighbourhood of Y, also follows from the previous identity. □

Recall that the conormal bundle $N^*_{Y|X}$ sitting in the exact sequence

$$0 \longrightarrow N^*_{Y|X} \longrightarrow \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0$$

is the bundle of 1-forms along Y which vanish in restriction to Y, or equivalently whose evaluation against vectors in $T_Y \subset T_X|_Y$ is zero. If we cover Y by (small) open sets U on which $Y \cap U = \{f_i^U = 0, i = 1 \cdots k\}$, then $N^*_{Y|X}|_U$ is trivialized by the df_i^U restricted to $Y \cap U$. If U, V are two overlapping sets, then the previous construction yields matrices M_{ji}^{UV} such that $f_i^U = \sum_j M_{ji}^{UV} f_j^V$. In particular,

$$df_i^U = \sum_j M_{ji}^{UV} df_j^V$$

holds in restriction to $Y \cap U \cap V$ so that the restrictions of M_{ji}^{UV} to Y are nothing but the transition matrices for $N^*_{Y|X}$ corresponding to the chosen cover.

To an open set U as before, associate

$$\widetilde{U}_Y := \{(z, Z = [Z_1 : \dots : Z_k]) \in U \times \mathbb{P}^{k-1} \,|\, Z_i f_j(z) = Z_j f_i(z), \forall i, j \leqslant k\},$$

called the blow up of U along Y. It is not difficult to see that $\widetilde{U}_Y$ is a smooth manifold and that the first projection $\pi : \widetilde{U}_Y \to U$ induces an isomorphism over $U \setminus Y$, whose inverse is given by $z \mapsto (z, [f_1(z) : \dots : f_k(z)])$. Moreover, $\pi^{-1}(Y) \simeq Y \times \mathbb{P}^{k-1}$.

Now, if we have two overlapping sets U, V as before, we obtain two manifolds $\widetilde{U}_Y, \widetilde{V}_Y$ with respective projections $\pi_U : \widetilde{U}_Y \to U$ and $\pi_V : \widetilde{V}_Y \to V$.

Lemma 6.12. *There exists a natural isomorphism*

$$\phi_{UV} : \pi_U^{-1}(U \cap V) \to \pi_V^{-1}(U \cap V)$$

such that $\pi_U = \pi_V \circ \phi_{UV}$.

Proof. Thee existence of ϕ_{UV} is clear outside Y, so it is enough to construct it locally in the neighborhood of $\pi_U^{-1}(Y \cap U \cap V)$ hence we can shrink U, V as needed. Set $P := P_{UV} := {}^t(M^{UV})^{-1}$ so that $f_j^V = \sum_i P_{ji} f_i^U$ for all j. Next, set $W = PZ$ viewed as column vectors of length k, i.e. $W_i = \sum_k P_{ik} Z_k$. Then we have for any indices i, j and $(z, Z) \in \pi^{-1}(U \cap V)$:

$$W_i f_j^V = \sum_k P_{ik} Z_k \sum_\ell P_{j\ell} f_\ell^U = \sum_{k,\ell} P_{ik} P_{j\ell} Z_k f_\ell^U = \sum_{k,\ell} P_{ik} P_{j\ell} Z_\ell f_k^U = \sum_\ell P_{j\ell} Z_\ell \sum_k P_{ik} f_k^U = W_j f_i^V,$$

that is, $(z, PZ) \in \pi_V^{-1}(U \cap V) \subset \widetilde{V}_Y$. Therefore, one can define $\phi_{UV}(z, Z) = (z, PZ)$ which is clearly invertible and commutes with the projections to $U \cap V$. □

Definition 6.13. If $Y \subset X$ is a submanifold, the local blow-ups $\widetilde{U}_Y$ for U an open covering of Y in X glue to a global manifold $\widetilde{X}_Y$ called the blow-up of X along Y. It comes equipped with a projection $\pi : \widetilde{X}_Y \to X$ which is an isomorphism away from Y.

Set $E := \pi^{-1}(Y)$. It follows from the local description of $\widetilde{X}_Y$ that the restriction map $\pi|_E : E \to Y$ is a $\mathbb{P}^{k-1}$-bundle; in particular, $E \subset \widetilde{X}_Y$ is a smooth hypersurface. Moreover, the transition matrices are the matrices $P_{UV} = {}^t(M^{UV})^{-1}$ introduced in the proof of Lemma 6.12 above. It follows from the discussion above that we have an isomorphism

$$E \simeq \mathbb{P}(N_{Y|X}) \tag{6.45}$$

of $\mathbb{P}^{k-1}$-bundles over Y. Moreover, we have the following.

Lemma 6.14. *The normal bundle $N_{E|\widetilde{X}_Y} \simeq \mathcal{O}_E(E)$ of $E \subset \widetilde{X}_Y$ is isomorphic to the tautological bundle $\mathcal{O}_{\mathbb{P}(N_{Y|X})}(-1)$ under the isomorphism* (6.45).

Proof. In the following, we write $\widetilde{X}$ for $\widetilde{X}_Y$ in order to lighten the notations. The differential of π induces a map $T_{\widetilde{X}} \to \pi^* T_X$ which we then restrict to E, yielding the morphism $f : T_{\widetilde{X}|_E} \to \pi^* T_X|_Y$. Since π send E to Y, we have, $f(T_E) \subset \pi^* T_Y$ so that f induces $\bar{f} : N_{E|\widetilde{X}} \to \pi^* N_{Y|X}$.

It is perhaps a bit clearer to denote by $V := N_{Y|X}$ the bundle over Y of rank $k-1$. Its pullback $\pi^* V$ to $E \simeq \mathbb{P}(V)$ fits in the cartesian square

$$\begin{tikzcd}
\pi^* V \arrow[r] \arrow[d] & V \arrow[d] \\
\mathbb{P}(V) \arrow[r, "\pi"] & Y
\end{tikzcd}$$

and admits the tautological subbundle $S := \mathcal{O}_{\mathbb{P}(V)}(-1) \subset \pi^* V$ defined by

$$S = \{([v], w) \in \mathbb{P}(V) \times V; v \in V \setminus \{0\}, w \in \mathbb{C}v\}.$$

Moreover, S has rank one. The lemma is a consequence of the fact that $\bar{f}$ induces an isomorphism $N_{E|\widetilde{X}} \to S$. This is now a local problem.

So one picks local coordinates $(z_1, \ldots, z_n)$ on an open set U centered around a given point $y \in Y$ such that $Y = (z_1 = \ldots = z_k = 0)$. In particular, $V|_{U \cap Y}$ is trivialized by the vector fields $\xi_i = \frac{\partial}{\partial z_i}$ for $i = 1, \ldots, k$. Next, one picks a point $x \in E$ such that $\pi(x) = y$.

Note that on $\tilde{U} := \pi^{-1}(U) \subset E$, one has $E|_{\tilde{U}} = \mathbb{P}(V)|_{\tilde{U}} \simeq U \times \mathbb{P}^{k-1}$. Without loss of generality, one can assume that $x = (0, [Z_1 : \ldots : Z_k])$ with $Z_1 \neq 0$. That is, we have identified $[Z] \in \mathbb{P}^{k-1}$ with the class of the vector field $\sum_{i=1}^{k} Z_i \xi_i$ in $\mathbb{P}(V|_U)$. Set $w_i = \frac{Z_i}{Z_1}$ for $i = 2, \ldots, k$ so that the functions $(z_1, w_2, \ldots, w_k, z_{k+1}, \ldots, z_n)$ yield a coordinate system on some open set near x in $\widetilde{X}$ under which the map π reads

$$\pi(z_1, w_2, \ldots, w_k, z_{k+1}, \ldots, z_n) = (z_1, z_1 w_2, \ldots, z_1 w_k, z_{k+1}, \ldots, z_n) \tag{6.46}$$

and such that E is given by the equation $(z_1 = 0)$. In particular, $N_{E|\widetilde{X}}$ is trivialized by the class $[\xi]$ of $\xi := \frac{\partial}{\partial z_1}|_E = \xi_1|_E$. Since

$$\pi_* \xi = \frac{\partial}{\partial z_1} + \sum_{i=2}^{k} w_k \frac{\partial}{\partial z_i} = \frac{1}{Z_1} \sum_{i=1}^{k} Z_i \xi_i$$

does not vanish and satisfies $(\pi_* \xi)_x \in S_x$, it follows that $\bar{f}$ induces an isomorphism onto its image S, as claimed. □

Corollary 6.15. *Let X be a compact manifold, let $\pi : \widetilde{X} \to X$ be the blow-up of a smooth submanifold $Y \subset X$ of codimension $k \geqslant 2$, and let E be the exceptional divisor. Given any positive line bundle $L \to X$, the line bundle $\pi^*(L^{\otimes m}) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is positive for $m \gg 1$. In particular, $\widetilde{X}$ is a Kähler manifold.*

Proof. Let h_X be an arbitrary hermitian metric on T_X; it induces by restriction to Y and quotient a metric on N_Y and then also a metric h on $\mathcal{O}_{\mathbb{P}(N_{Y|X})}(1)$. For any $y \in Y$, let $E_y := \mathbb{P}(N_{Y,y}) = \pi^{-1}(y)$. The hermitian line bundle $(\mathcal{O}_{\mathbb{P}(N_{Y|X})}(1), h)|_{E_y}$ is isometric to $(\mathcal{O}_{\mathbb{P}^{k-1}}(1), h_{\mathrm{FS}})$ (cf Remark 4.1) hence its Chern curvature form is definite positive.

From Proposition 6.5, $\mathcal{O}_{\widetilde{X}}(-E)|_E \simeq N^*_{E|\widetilde{X}} \simeq \mathcal{O}_E(1)$, hence we can extend h arbitrarily to a metric h_E on $\mathcal{O}_{\widetilde{X}}(-E)$ whose Chern form $\Theta_E := \Theta_h(\mathcal{O}_{\widetilde{X}}(-E))$ is positive along E (i.e. in restriction to E, in the directions of E).

Let h_L be a hermitian metric on L with positive Chern form, and let $F := \pi^*(L^{\otimes m}) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ be endowed with the metric $h_F := \pi^* h_L^{\otimes m} \otimes h_E$. We have for any $v \in T_{\widetilde{X}}$

$$\Theta_{h_F}(F)(v, \bar{v}) = m\Theta_{h_L}(d\pi(v), \overline{d\pi(v)}) + \Theta_E(v, \bar{v}).$$

Write a C^∞ decomposition $T_{\widetilde{X}}|_E = V \oplus W$ where $V = \ker(d\pi)$ and $W = V^\perp$ so that $d\pi$ induces a isomorphism $W \to \pi^* T_Y$. For $v \in V$, we have $\Theta_E(v, v) \geqslant c|v|^2$ while for $w \in W$, we have $\Theta_{h_L}(d\pi(w), \overline{d\pi(w)}) \geqslant c'|w|^2$, for some positive constants c, c'. By Cauchy-Schwarz, it follows that $\Theta_{h_F}(F)(v, \bar{v}) \geqslant c''|v|^2$ for any $v \in T_{\widetilde{X}}|_E$ up to taking m large enough. By continuity of $\Theta_{h_F}(F)$ on the unitary tangent bundle of $\widetilde{X}$, we see that there exists an open neighborhood U of E such that

$$\Theta_{h_F}(F)(v, \bar{v}) \geqslant \frac{1}{2} c''|v|^2 \quad \forall v \in T_{\widetilde{X},x}, x \in U.$$

On the complement of U, π is an isomorphism hence $\pi^* \Theta_{h_L}$ is a positive $(1,1)$-form there. By compactness of $\widetilde{X} \setminus U$, there exists m such that $m\pi^* \Theta_{h_L} + \Theta_E$ is still positive on that set. This implies that $\Theta_{h_F}(F)$ is a Kähler form on the whole $\widetilde{X}$ for m large enough. □

Corollary 6.16. *Let X be a compact manifold and let $\pi : \widetilde{X} \to X$ be a composition of blow-up with smooth centers and let $E = \sum_{i=1}^{N} E_i$ be the exceptional divisor. Given any positive line bundle $L \to X$, there exist positive numbers $a_1, \ldots, a_N$ such that the line bundle $\pi^*(L^{\otimes k}) \otimes \mathcal{O}_{\widetilde{X}}(-\sum a_i E_i)$ is positive for $k \gg 1$.*

Proof. In order to simplify the notations, we treat the case of two successive blow-ups

$$\widetilde{X} = X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X.$$

Let F_1 (resp. F_2) be the exceptional divisor of π_1 (resp. π_2). The exceptional divisor $E = E_1 + E_2$ of π has too components: E_1 is the strict transform of F_1 by π_2 while $E_2 = F_2$. Note that $\pi_2^* F_1 = E_1 + bE_2$ where $b \geqslant 0$ is positive if and only if F_1 intersects the center of π_2.

By Proposition 6.15, $\pi_1^* L^{k_1} \otimes \mathcal{O}_{X_1}(-F_1)$ is positive for some large $k_1 > 0$. By the same token, $\pi_2^*(\pi_1^* L^{k_1} \otimes \mathcal{O}_{X_1}(-F_1))^{\otimes k_2} \otimes \mathcal{O}_{X_2}(-E_2)$ is positive for some large $k_2 > 0$. But that bundle is nothing but $\pi^*(L^{k_1+k_2}) \otimes \mathcal{O}_{\widetilde{X}}(-k_2 E_1 - (1+bk_2)E_2)$.

Finally, for $k \geqslant k_1 + k_2$, one sets $k' := k - (k_1 + k_2) \geqslant 0$ and $a_1 := k_2, a_2 := 1 + bk_2$. Then, the decomposition

$$\pi^*(L^{\otimes k}) \otimes \mathcal{O}_{\widetilde{X}}(-\sum a_i E_i) = \pi^*(L^{\otimes k'}) \otimes \left(\pi^*(L^{k_1+k_2}) \otimes \mathcal{O}_{\widetilde{X}}(-\sum a_i E_i)\right),$$

yields the result since the sum of a semipositive form and a positive form is positive. $\square$

Lemma 6.17. *Let $\pi : \widetilde{X} \to X$ be the blow up of a smooth submanifold $Y \subset X$ of codimension $k \geqslant 1$. Then we have*

$$K_{\widetilde{X}} \simeq \pi^* K_X + (k-1)E$$

where $E \subset \widetilde{X}$ is the exceptional divisor of π.

Proof. The differential of π induces a map $T_{\widetilde{X}} \to \pi^* T_X$. Dualizing, we get a map $\pi^*\Omega_X^1 \to \Omega_{\widetilde{X}}^1$. Taking the n-th exterior power, we get a map $\pi^* K_X \to K_{\widetilde{X}}$, hence a holomorphic section s of the line bundle $K_{\widetilde{X}} \otimes \pi^* K_X^{-1}$. This section does not vanish away from E since π is an isomorphism there. Using e.g. Lemma 3.18, our present lemma is then equivalent to the assertion that s vanishes at order exactly $k-1$ along E. This can now be checked by a local computation.

Near a given point $x \in E$, one can find an open set U endowed with coordinates $(z_1, \ldots, z_n)$ such that $E = (z_1 = 0)$ on our chart and

$$\pi(z) = (z_1, z_1 z_2, \ldots, z_1 z_k, z_{k+1}, \ldots, z_n),$$

cf (6.46). Denote by $V \subset \mathbb{C}^n$ an open set such that $\pi(U) \subset V$. On U (resp. V), we have a trivialization $\tau := dz_1 \wedge \ldots \wedge dz_n$ of K_U (resp. $\sigma := dw_1 \wedge \ldots \wedge dw_n$ of K_V) and s is nothing but the ratio $\frac{\pi^*\sigma}{\tau}$. Given the expression of π above, an straightforward computation shows that $\pi^*\sigma = z_1^{k-1}\tau$, ending the proof of the lemma. $\square$

Lemma 6.18. *Let X be a compact complex manifold and let L be a line bundle on X. Let $\pi : \widetilde{X} \to X$ be the blow-up of a submanifold of codimension at least two and let E be the exceptional* *divisor. Finally, let $s_E \in H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(E))$ be a section cutting out E. Then for any non-negative integer $m \geqslant 0$, the map*

$$H^0(X, L) \longrightarrow H^0(\widetilde{X}, \pi^* L \otimes \mathcal{O}_{\widetilde{X}}(mE))$$

sending s to $\pi^* s \otimes s_E^{\otimes m}$ is an isomorphism.

Proof. Clearly, this map is injective. Now, let $\tau \in H^0(\widetilde{X}, \pi^* L \otimes \mathcal{O}_{\widetilde{X}}(mE))$. Consider the meromorphic section $\sigma := \frac{\tau}{s_E^{\otimes m}}$ of $\pi^* L$. The restriction $\sigma|_{\widetilde{X} \setminus E}$ is holomorphic and since $\pi|_{\widetilde{X} \setminus E} : \widetilde{X} \setminus E \to X \setminus Y$ is an isomorphism, we get a section $s \in H^0(X \setminus Y, L)$ such that $\sigma|_{\widetilde{X} \setminus E} = \pi^* s$. In other words, we have $\tau = \pi^* s \otimes s_E^{\otimes m}$ over $\widetilde{X} \setminus E$. By Hartogs, s actually extends across Y to a holomorphic section of L on X, hence we get $\tau = \pi^* s \otimes s_E^{\otimes m}$ everywhere on X, hence the lemma. □

7 Geometric application of Kodaira's vanishing theorem

7.1 Kodaira's vanishing theorem

We recall the following fundamental theorem:

Theorem 7.1 (Kodaira vanishing theorem). *Let X be a compact complex manifold and let L be a positive line bundle on X. We have*

$$H^q(X, K_X + L) = 0 \quad \forall q > 0.$$

More generally, one has

$$H^q(X, \Omega_X^p \otimes L) = 0 \quad \text{if } p + q > n.$$

A few remarks before going any further.

(a) In this section and from now on, we will use the additive notation for the tensor product on line bundles. E.g. $K_X + L$ corresponds to $K_X \otimes L$.

(b) Recall that a line bundle L is said to be positive if it admits a hermitian metric h such that its Chern curvature form $\Theta_h(L)$ is a positive $(1,1)$-form. In particular, the latter is a Kähler form and X is a Kähler manifold.

(c) The above vanishing can be interpreted in coherent cohomology or in Dolbeault cohomology. In the latter case, it is equivalent to saying that any given any L-valued form α of type (n, q) with $q > 0$ such that $\bar{\partial}\alpha = 0$, there exists an L-valued form β of type $(n, q-1)$ such that $\alpha = \bar{\partial}\beta$.

We will sometimes also use the following variant of Kodaira vanishing known as Serre vanishing, which can be obtained in a similar fashion.

Theorem 7.2 (Serre vanishing). *Let L be a positive line bundle on a compact complex manifold. There for any vector bundle E on X, there exists $m_0 \geqslant 0$ such that*

$$\forall q > 0, \forall m \geqslant m_0, \quad H^q(X, E \otimes L^{\otimes m}) = 0.$$

7.2 Cohomology of Fano manifolds

An important application of Kodaira's vanishing concerns Fano manifolds, which are compact Kähler manifolds X such that $-K_X$ is positive.

Corollary 7.3. *Let X be a Fano manifold of dimension n. Then*

$$\forall q > 0, \quad H^q(X, \mathcal{O}_X) = H^0(X, \Omega_X^q) = 0.$$

Proof. Write $\mathcal{O}_X = K_X \otimes K_X^{-1}$ and apply Theorem 7.1. The equality between the two cohomology spaces follows from Hodge duality. $\square$

Let us now focus more specifically on the projective space, where much more can be said.

Proposition 7.4. *Let $n \geqslant 1$ be an integer. We set $d(n,k) = \binom{n+k}{n} = \dim S^k\mathbb{C}^{n+1}$. We have*

$$h^q(\mathbb{P}^n, \mathcal{O}(k)) = \begin{cases} d(n,k) & \text{if } q = 0,\ k \geqslant 0, \\ d(n, -k-n-1) & \text{if } q = n,\ k \leqslant -(n+1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first two cases can be seen using Proposition 4.2, Serre duality $H^n(\mathbb{P}^n, \mathcal{O}(k)) \simeq H^0(\mathbb{P}^n, K_{\mathbb{P}^n}(-k))^*$ and the isomorphism $K_{\mathbb{P}^n} \sim \mathcal{O}(-n-1)$, see Corollary 6.3. If $q = 0$ (resp. $q = n$) and $k < 0$ (resp. $k > -(n+1)$), then one sees similarly that $h^q(\mathcal{O}(k)) = 0$. Now, if $q \neq 0, n$, Kodaira vanishing yields $h^q(\mathcal{O}(k)) = h^q(K_{\mathbb{P}^n}(k+n+1))$ if $k > -(n+1)$. If $k \leqslant -(n+1)$, we use Serre duality to get $h^q(\mathcal{O}(k)) = h^{n-q}(K_{\mathbb{P}^n}(-k))$ which vanishes since $-k > 0$, thanks to Kodaira vanishing again. $\square$

Proposition 7.5. *The tangent bundle of $\mathbb{P}^n$ satisfies the following properties:*

1. *The space of global holomorphic vector fields is given by*

   (7.47) $$H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) \simeq V \otimes V^* / \mathbb{C}\mathrm{Id}_V$$

   where $V = H^0(\mathbb{P}^n, \mathcal{O}(1))$ and where we have identified $V \otimes V^*$ with $\mathrm{End}(V)$.

2. *There is an isomorphism*

   $$H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) \simeq H^{n,n}(\mathbb{P}^n) \simeq \mathbb{C}$$

   In particular, the Griffiths positive bundle $T_{\mathbb{P}^n}$ is not a Nakano positive vector bundle.

3. *$T_{\mathbb{P}^2}$ is not isomorphic to a direct sum of two line bundles.*

Proof. The first item follows from Proposition 7.4 and the Euler exact sequence (4.26).

As for the second item, remember that $H^q(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes \mathcal{O}(1)) = 0$ for $q > 0$ by Kodaira vanishing. Therefore, the long exact sequence in cohomology one gets after tensoring (4.26) with $K_{\mathbb{P}^n}$ gives

$$H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes \mathcal{O}(1)) = 0 \to H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) \to H^n(\mathbb{P}^n, K_{\mathbb{P}^n}) \to H^n(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes \mathcal{O}(1)) = 0$$

hence the result.

Let us get on to the last assertion. Arguing by contradiction, write $T_{\mathbb{P}^2} = \mathcal{O}(a) \oplus \mathcal{O}(b)$ with $a \geqslant b$, cf (4.3). Taking the determinant, we see that one must have $a + b = 3$. From the first item we see that

$$h^0(\mathbb{P}^2, \mathcal{O}(a)) + h^0(\mathbb{P}^2, \mathcal{O}(b)) = 8.$$

In particular, one must have $a > 0$. If $b \leqslant 0$, then $a \geqslant 3$ and but $h^0(\mathbb{P}^2, \mathcal{O}(a)) = \binom{2+a}{2} \geqslant \binom{5}{2} = 10 > 8$, cf Proposition 7.4. Therefore the only combination left is $(a, b) = (2, 1)$. Since $\binom{4}{2} + \binom{3}{2} = 9$, this is not possible either. □

7.3 Lefschetz hyperplane theorem for cohomology

Theorem 7.6 (Weak Lefschetz theorem)**.** *Let $Y \subset X$ be a smooth hypersurface in a compact Kähler manifold X of dimension n such that $\mathcal{O}_X(Y)$ is positive. Then the canonical restriction map*

$$H^k(X, \mathbb{C}) \longrightarrow H^k(Y, \mathbb{C})$$

is bijective for $k \leqslant n-2$ and injective for $k \leqslant n-1$.

Proof. For any integers $0 \leqslant p, q \leqslant n-1$, we have natural restriction maps $H^{p,q}(X) \longrightarrow H^{p,q}(Y)$. Moreover, the Hodge decompositions of $H^{p+q}(X, \mathbb{C})$ and $H^{p+q}(Y, \mathbb{C})$ are compatible with the said maps. Therefore it is sufficient to prove that $H^q(X, \Omega_X^p) \to H^q(Y, \Omega_Y^p)$ is bijective for $p + q \leqslant n-2$, and injective for $p + q = n-1$.

By tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-Y) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

by Ω_X^p, we get

$$0 \longrightarrow \Omega_X^p(-Y) \to \Omega_X^p \to \Omega_X^p|_Y \longrightarrow 0.$$

Now Serre duality and Kodaira vanishing yield

$$H^q(X, \Omega_X^p(-Y)) = H^{p,q}(X, \mathcal{O}_X(Y)) \simeq H^{n-q,n-p}(X, \mathcal{O}_X(Y))$$

which vanishes for $p + q < n$. In particular, the induced map in cohomology

$$H^q(X, \Omega_X^p) \longrightarrow H^q(Y, \Omega_X^p|_Y) \tag{7.48}$$

is bijective for $p + q < n-1$ and injective for $p + q = n-1$. Next, taking the p-th exterior power of the exact sequence (6.42), one obtains the following exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-Y) \otimes \Omega_Y^{p-1} \to \Omega_X^p|_Y \to \Omega_Y^p \longrightarrow 0.$$

Indeed, the wedge product induces a map $N_{Y|X}^* \otimes \Omega_Y^{p-1} \to \Omega_X^p|_Y$ and it is clear (e.g. by taking local coordinates adapted to Y) that this map is injective with image the kernel of the restriction map $\Omega_X^p|_Y \to \Omega_Y^p$. Similarly to what we explained above, we have that

$$H^q(Y, \Omega_Y^{p-1}(-Y)) \simeq H^{n-1-q,n-p}(Y, \mathcal{O}_Y(Y))$$

vanishes for $p+q<n$. Therefore, the induced map

$$H^q(Y,\Omega^p_X|_Y) \longrightarrow H^q(Y,\Omega^p_Y) \tag{7.49}$$

is bijective for $p+q<n-1$ and injective for $p+q=n-1$. Since the restriction map $H^q(X,\Omega^p_X) \to H^q(Y,\Omega^p_Y)$ is the composition of the two maps (7.49) and (7.48), the result follows. □

Here are a few applications/comments:

1. The only complex tori that can be embedded as a complete intersection in the projective space are elliptic curves.
2. An hypersurface in $\mathbb{P}^n$ with $n \geqslant 3$ does not admit any non-zero one forms.
3. Show that the bounds are sharp (e.g. look at H^2 of a quartic surface).
4. One can deduce Kodaira's vanishing theorem from Theorem 7.6 using cyclic covers, cf [Laz04, Chapter 4]. Of course, one can prove an even stronger version of Theorem 7.6 without relying on Kodaira's vanishing theorem, but rather on Morse theory.

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